PRACTICAL AND THEORETICAL GEOMETRY FOR SCHOOLS

BY

H. ARMITSTEAD, B.Sc.

PRIZEMAN, VICTORIA UNIVERSITY OF MANCHESTER, AND TEMPORARY PROFESSOR OF MATHEMATICS, THE PRESIDENCY COLLEGE, CALCUTTA

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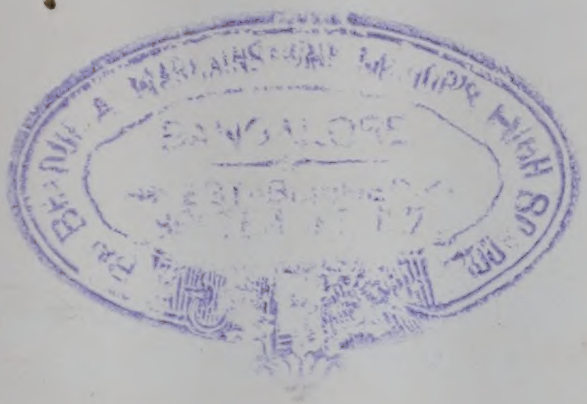

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PREFACE

THIS book has been prepared to provide an introduction to Practical and Theoretical Geometry for those who are just commencing the subject. It will be specially helpful to the students who are preparing for the Calcutta University Matriculation Examination. Though it has been written with a view of meeting the new requirements of the Calcutta University, it will be found to be just as suitable for the Madras University Matriculation Examination, the Oxford and Cambridge Locals, etc.

An attempt has been made in writing the book to make the subject a little more practical and interesting than is usually found to be the case in elementary books on Geometry.

The innovation is the prominence given to the logical arrangements of the exercises. Great attention has been paid to the advisability of treating the subject inductively as far as possible. The pupils are led to discover geometrical truths which they prove later. Each theorem is introduced by experimental work, so that the pupils may wholly realize the truths dealt with.

Part I. of the Geometry includes roughly the whole of Euclid Book I.

Part II. is divided into Books II. and III. In Book II. the order of the theorems agrees generally with the order of those of Euclid Book II., but differs very much from them in

treatment through the use of hypothetical constructions. Book III. contains the gist of Euclid Book III., which has had a number of unnecessary propositions deleted from it.

As in Part I., prominence has been given to the logical arrangement of exercises, and to the introduction of theorems by experimental work, so that the geometrical truths may be fully realized.

The Author will welcome any corrections.

H. ARMITSTEAD.

PRESIDENCY COLLEGE, CALCUTTA,
September, 1908.

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INTRODUCTION

Necessary Instruments.

1. A ruler graduated in inches and tenths of an inch, and also in centimetres and millimetres.

2. Set squares; one with angles of 60° and 30°, another with angles of 45°.

3. A semicircular or rectangular protractor.

4. A pair of compasses.

5. Tracing-paper. [Too much stress cannot be laid upon the value of tracing-paper.]

6. A pair of dividers.

7. A hard pencil.

8. A pair of scissors.

Throughout this course geometrical truths are dealt with. These truths should first be realized by measurement and observation. A full record of such work should be kept in the scholars' large drawing-books. The pages of the books should be divided into **six** sections. The figures should be clearly drawn and lettered. *The conclusions arrived at should be written underneath the figures.*

Ideas of dimensions and points, lines and surfaces, should first be obtained from consideration and observation of simple solids, such as cubes, cones, etc.

Definitions and axioms should only be introduced when required. Propositions and exercises should be introduced by experimental work, in order that the pupils should fully realize the truths dealt with.

The proof of a proposition should **immediately** follow the practical exercises bearing upon it.

SYLLABUS

PRACTICAL.

Bisection of angles and straight lines.
Construction of perpendiculars to straight lines.
Construction of an angle equal to a given angle.
Construction of parallels to a given straight line.
Construction of triangles with given parts.
Division of a straight line into a given number of equal parts.
Construction of a parallelogram equal to a given triangle, and having one of its angles equal to a given angle.
Construction of a triangle equal in area to a given rectilineal figure.
Construction of a tangent to a circle.
Easy extensions of these constructions may be given as problems.
Candidates may be required to give the reasons for any particular construction involved in any question.

THEORETICAL.

Angles at a Point.

If a straight line stands on another straight line, the sum of the two angles so formed is equal to two right angles; and the converse.
If two straight lines intersect, the vertically opposite angles are equal.

Parallel Straight Lines.

If a straight line, cutting two other straight lines, makes—
(1) the alternate angles equal,
(2) two corresponding angles equal,
(3) the interior angles on the same side of the line supplementary,
then the two straight lines are parallel; and the converse.
Straight lines which are parallel to the same straight line are parallel to one another.

Triangles and Rectilineal Figures.

The sum of the angles of a triangle is equal to two right angles.
If the sides of a convex polygon are produced in order, the sum of the angles so formed is equal to four right angles.
Two triangles are equal in every respect—
(1) If two sides and the included angles of one triangle are respectively equal to two sides and the included angle of the other.
(2) If two angles and a side of the one triangle are respectively equal to two angles and the corresponding side of the other.

If two sides of a triangle are equal, the angles opposite to these sides are equal; and the converse.

Two triangles are equal in every respect if the three sides of one triangle are respectively equal to the three sides of the other.

Two right-angled triangles are equal in every respect if they have their hypotenuses equal, and one side of the one equal to one side of the other.

If two sides of a triangle are unequal, the greater side has the greater angle opposite to it; and the converse.

Any two sides of a triangle are greater than the third.

Of all the straight lines that can be drawn to a given straight line from a given point without it, the perpendicular is the shortest.

The opposite sides and angles of a parallelogram are equal; each diagonal bisects the parallelogram, and the diagonals bisect one another.

If there are three or more parallel straight lines, and the intercepts made by them on any straight line that cuts them are equal, then the corresponding intercepts on any other straight line that cuts them are equal.

Areas.

Parallelograms on the same or equal bases, and of the same altitude, are equal in area.

Triangles on the same or equal basis and of the same altitude are equal in area.

Equal triangles on the same or equal bases are of the same altitude.

Illustrations and explanations of the geometrical theorems corresponding to the following algebraical identities—

$$k(a+b+c)\ \ldots = ka + kb + kc\ \ldots$$
$$(a+b)^2 = a(a+b) + b(a+b)$$
$$a(a+b) = a^2 + ab$$
$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a-b)^2 = a^2 - 2ab + b^2$$
$$a^2 - b^2 = (a+b)(a-b)$$

The square on a side of a triangle is greater than, equal to, or less than the sum of the squares on the other two sides, according as the angle contained by those sides is obtuse, right, or acute. The difference in the cases of unequality is twice the rectangle contained by one of the two sides, and the projection on it of the other.

Loci.

The locus of a point which is equidistant from two fixed points is the perpendicular bisector of the straight line joining the two fixed points.

The locus of a point which is equidistant from two intersecting straight lines consists of the pair of straight lines which bisect the angles between the two given lines.

The Circle.

A straight line drawn from the centre of a circle to bisect a chord which is not a diameter is at right angles to the chord; conversely, the perpendicular to a chord from the centre bisects the chord.

There is one circle, and one only, which passes through three given points not in a straight line.

In equal circles (or in the same circle) (1) if two arcs subtend equal angles at the centre they are equal; (2) conversely, if two arcs are equal, they subtend equal angles at the centre.

In equal circles (or in the same circle) (1) if the chords are equal they cut off equal arcs; (2) conversely, if two arcs are equal, the chords of the arcs are equal.

Equal chords of a circle are equidistant from the centre; and the converse.

The tangent at any point of a circle is perpendicular to the radius through the point.

If two tangents are drawn to a circle from an external point (1) they are equal; (2) they subtend equal angles at the centre of the circle.

If two circles touch, the point of contact lies on the straight line through the centres.

The angle which an arc of a circle subtends at the centre is double that which it subtends at any point on the remaining part of the circumference.

Angles in the same segment of a circle are equal; and if the line joining two points subtends equal angles at two other points on the same side of it, the four points lie on a circle.

The angle in a semicircle is a right angle; the angle in a segment greater than a semicircle is less than a right angle; and the angle in a segment less than a semicircle is greater than a right angle.

The opposite angles of any quadrilateral inscribed in a circle are supplementary; and the converse.

If a straight line touch a circle and from the point of contact a chord be drawn, the angles which this chord makes with the tangent are equal to the angles in the alternate segments.

If two chords intersect either inside or outside the circle, the rectangle contained by the parts of the one is equal to the rectangle contained by the parts of the other.

Each question on Theoretical Geometry shall consist of a theorem contained in the above schedule together with an easy deduction.

Any proof of a proposition shall be accepted, which appears to the Examiners to form part of a systematic treatment of the subject, but proofs of theorem should, as far as possible, be based on first principles; the order in which the theorems are stated in the above schedule is not to be regarded as essential.

In the proof of theorems and deductions from them, it shall be permissible to use hypothetical constructions.

The ordinary symbolical abbreviations may be used.

The above syllabus is prescribed for the Calcutta University Matriculation Examination. To meet the requirements of the Madras University Matriculation Examination, the following theorems have been added :—

Areas.

Illustrations and explanations of the geometrical theorems corresponding to the following algebraical identities—

(1) $(a+b)^2-(a-b)^2=4ab.$

(2) $(a+b)^2+(a-b)^2=2a^2+2b^2.$

Loci.

(3) The locus of the vertices of all triangles, which have the same base and the sum of the squares of their sides equal to a given square, is a circle having its centre at the middle point of the base.

(4) The locus of the vertices of all the triangles, which have the same base and the difference of the squares of their sides equal to a given square is a straight line perpendicular to the base.

(5) The locus of the vertices of all the triangles, which have the same base and their vertical angles equal to a given angle, is the arc of a segment of a circle.

PRACTICAL AND THEORETICAL GEOMETRY.

BOOK I

Exercises on Straight Lines.

1.

A

The path traced out by the point A is

2.

A B C D

3.

A B C D E

AB = cm.
BC = cm.
CD = cm.
DE = cm.

AB+BC+CD+DE = cm.
Length of AE = cm.
Error = cm.

4.

A B C D E F

AB = cm.
BC = cm.
CD = cm.
DE = cm.
EF = cm.

AB+BC+CD+DE+EF = cm.
Length of AF = cm.
Error = cm.

5.

B

A

My ruler is

6.

B

A

The freehand is

Lines.

Practical Exercises on Straight Lines.

(1) Draw the path traced out by any point when moved 2″ in any direction.

What does the point trace out?

(2) Take four points, A, B, C, D, in order, and, as far as you can judge, in the same straight line.

Find out whether they are in a perfect straight line by measuring the lengths of AB, BC, CD, AD, and seeing whether AD is equal to the sum of the lengths AB, BC, CD, AD.

(3) Draw any straight line of not less than 2″; divide it unequally into four parts.

Measure the parts, and tabulate your results.

Test the correctness of your result by measuring the whole line.

(4) Repeat the last exercise, but dividing the line into five unequal parts.

(5) Test the accuracy of your straightedge or ruler.

Draw a straight line AB with your ruler; quite close to the line place your ruler. If *no* space is seen between the ruler and the line, the edge of the ruler is straight.

Second method.—Take a piece of tracing-paper and make a trace of one part of the line on it, and apply it to any other part in every possible way, even when the tracing-paper is inverted.

(6) Draw a freehand line and test its straightness.

N.B.—In ruling lines with your ruler always draw from left to right.

Exercises.

(1) Measure in inches and in centimetres the length of a page of your book.

(2) Measure the length and thickness of the desk at which you are working.

(3) Draw a line some five inches long and divide it into centimetres.

(4) Draw a line; measure its length in inches and centimetres. From the two results find how many centimetres there are in an inch.

(5) Find by measurement the actual distance between Calcutta and Madras, Calcutta and Bombay, in your atlas, following the scale given.

Surfaces.

Practical Exercises on Surfaces.

(1) Draw the figure traced out by a line, AB, 3 cm. long, when moved 5 cm. in any direction.

What does the line trace out?

(2) The line AB is rotated about the fixed point A.
What does the line trace out?

(3) Draw two intersecting lines, AB, CD, cutting one another at E; join AC, CB, BD, DA.
What do these lines bound?

(4) Take any three points A, B, and C, and join them.
What do the lines enclose?

(5) Test whether the surface of your desk is a plane surface.
Place the edge of your ruler against the surface and see if the edge touches the surface all along its length; if it does so in every position the surface is plane.

(6) Is the surface of your setsquare flat?

Exercises.

(1) Test the flatness of your watch.

(2) Prove by experiment that one of the walls in your room is a plane surface.

(3) See if your reading book is flat.

(4) Is your setsquare perfectly flat?

(5) Test the flatness of the upper and lower faces of your instrument box.

Angles.

Practical Exercises on Angles.

(1) Construct a rectangular or circular protractor the exact size of the one you use.

(2) Make an angle of 45°—
- (*a*) Using setsquare.
- (*b*) Using protractor.

(3) Make an angle of 135°—
- (*a*) Using setsquares.
- (*b*) Using protractor.

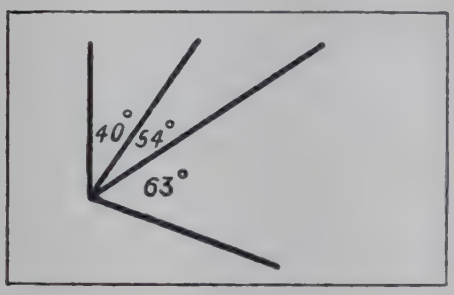
(4) Draw angles of 45°, 75°, 116°, 124°, 180°, with common apex and common arm.

(5) Construct a right angle—
(*a*) Using setsquare.
(*b*) Using protractor.

(6) Prove that right angles are equal.
Draw two right angles ABC, DEF ; make a tracing of the right angle DEF on tracing-paper, and apply it to the right angle ABC. See if the figures coincide, and state your conclusion.

Exercises.

(1) Draw two straight lines cutting one another in any position ; measure all the angles ; write down their magnitudes.



(2) With your protractor make the following angles :—
70°, 40°, 33°, 54°, 61°, 127°.

(3) Place the angles 40°, 54°, 63° adjacent to each other, and measure the whole angle formed. Verify the result by measuring the whole angle.

(4) Place the angles 60°, 35°, 45°, 24° adjacent to each other.

Practical Exercises on Angles.

(1) From a point A draw any three straight lines to form two angles. Write down—
(*a*) The vertex.
(*b*) The common arm.
(*c*) The value of each angle.
(*d*) The name of the angles.

(2) From a point C, in a straight line AB, draw any other straight line CD ; measure the adjacent angles.

(3) Draw a straight line CD to stand on a straight line AB : measure the angles ; find their sum.
[Angle BCD is the supplement of angle ACD.]

(4) Draw an angle of 45°, and then draw its supplement, but not adjacent to it, making the arms 2″ long.

(5) Draw a right angle ABC ; from B draw any line BD ; measure the angles now formed ; find their sum.
[Angle ABD is the complement of the angle CBD.]

(6) Draw two straight lines AB, CD intersecting at E ; measure all the angles. Write down the vertically opposite angles.

Exercises.

(1) From a point A make the two angles 47° and 54° adjacent to each other.

(2) Draw an angle of 65°; then draw its supplement, but not adjacent to it, making the arms 4·7 cm. long.

(3) Draw two straight lines cutting one another in any position; measure all the angles.

(4) Construct an angle of 34°; then draw its complement not adjacent to it, making the arms 2·4″ long.

Definitions.

A **point** is that of which there is no part.

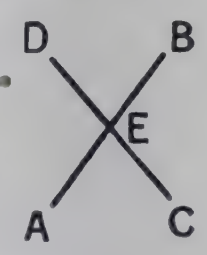
A **point** is that which has position but no magnitude.

A **line** is that which has length but no breadth.

If a point moves it traces out a line. Thus it can be seen from the definition of a point that a line has no breadth or thickness; it has only length.

The **extremities** of lines are points.

When the two extremities of a line are fixed, the line is said to be **finite**, or **limited**; when one or both extremities are not fixed, the line is said to be **infinite**, or **unlimited**.



Lines cut one another in points, unless they overlap. The line AB cuts the line DC at the point E.

Properties of a straight line—

(*a*) It is fixed by two points; *i.e.* if two points of it are fixed.

(*b*) It is fixed by a point and direction.

(*c*) It has the same direction at all points.

The following will serve as definitions of a **straight line**:—

(1) *A straight line is one such that only one of its kind can pass through two given points.*

(2) A straight line is said to be one of unvarying direction.

(3) A straight line is the shortest distance between two points.

From (1) note that—

(*a*) *Two straight lines cannot intersect at more than one point.*

(*b*) *If two straight lines coincide at two points they must coincide throughout.*

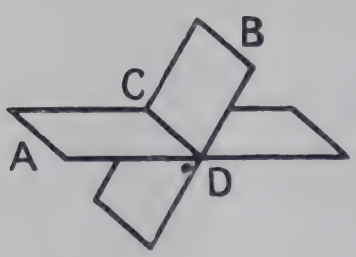
If a line be not straight it is called a **curved** line. A curved line is one of varying direction. A line made up of a number of straight lines in different directions is *not* to be confounded with a curved line.

A line equals a straight line in cases where no misunderstanding can arise.

A **surface** is that which has length and breadth, but not thickness.

A surface may also be defined as the boundary of a solid figure, or the boundary between two divisions of space.

If a line moves in space it traces out a surface.



The **boundaries** of a surface are lines.

A **boundary** is that which is the limit of anything.

Two surfaces intersect in a line, *e.g.* the planes **A** and **B** intersect in the straight line **CD**.

Properties of a plane surface—

(*a*) It is fixed by a straight line and a point.

(*b*) It is fixed by two intersecting straight lines.

(*c*) It is fixed by three points.

A **plane surface** is one such that only one of its kind can pass through three given points, or

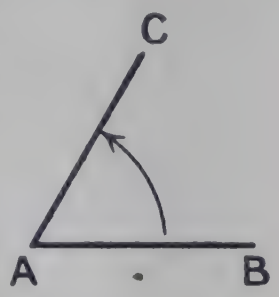
A plane surface is one in which any two points being taken, the straight line between them lies wholly on that surface.

If the straight line be produced indefinitely on both sides, it will lie wholly on that surface.

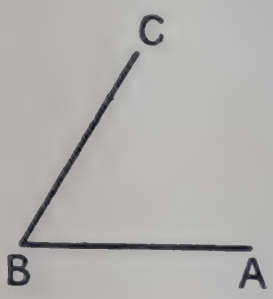
A plane surface is often briefly referred to as a **plane**.

If a surface be not plane, it is said to be **curved.**

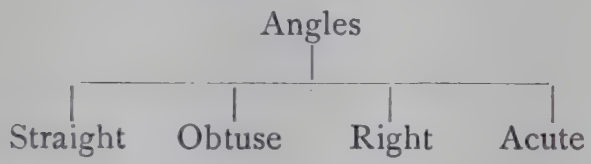
Suppose a straight line to start from the position AB, and to rotate about the point A in the direction shown by the arrowhead till it occupies the position AC; then the rotating line is said to have traced out the angle BAC.



An **angle** is the inclination of two straight lines, which meet together, but are not in the same straight line.

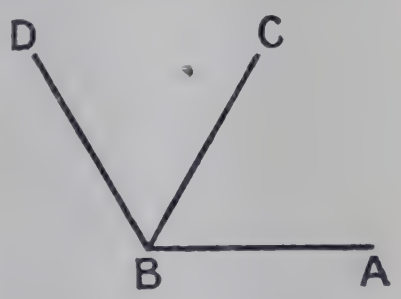


The point at which straight lines meet is called the **vertex** of the angle; the straight lines themselves, AB, BC, the **arms** of the angle.



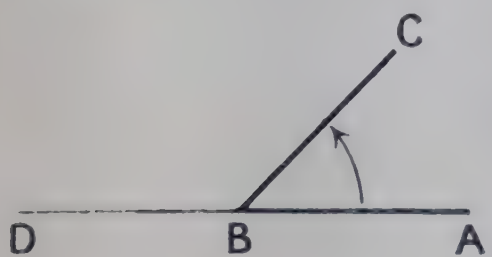
When the lines enclosing the angle are straight, the angle is called a **rectilinear angle.**

Two angles are said to be **adjacent** when they have a common vertex, a common arm, and do not overlap.



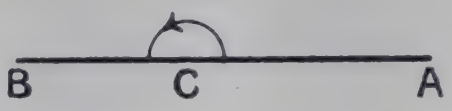
The angles ABC, CBD are adjacent, having B as common vertex and BC as common arm.

Suppose a straight line to start from the position BA, and to rotate about the fixed point B in the direction shown by the arrowhead until it lies in the position BD, where it is straight with its initial position, but in the opposite direction, then the angle traced out by the rotating line is a straight angle.

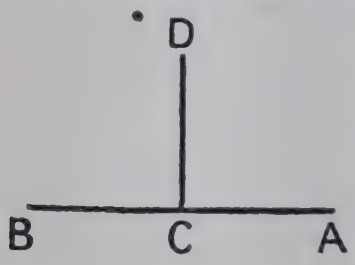


If the two arms of an angle lie in the same straight line

then the angle formed by them on either side is called a **straight angle.**



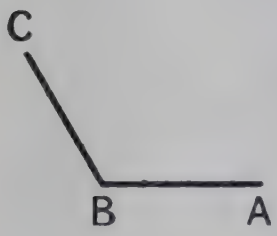
A straight angle is equal to two right angles.



When a straight line standing on another straight line makes the adjacent angles equal to one another, each of the angles is a **right angle.** The straight line which stands on the other is called a **perpendicular** to it.

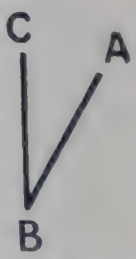
A right angle is half a straight angle.

All right angles are equal, as may be proved by superposition.

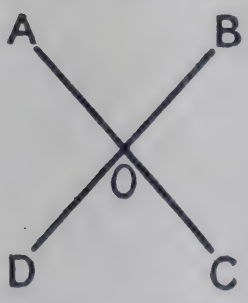


An **obtuse angle** is an angle greater than a right angle, but less than two right angles.

An **acute angle** is an angle which is less than a right angle.



If two angles be together equal to two right angles, they are said to be **supplementary,** *i.e.* each is *supplementary* to the other.



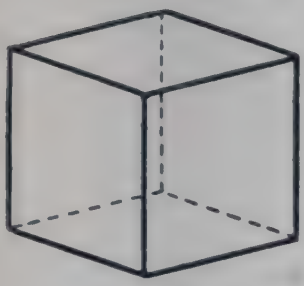
If two angles be together equal to a right angle, they are said to be **complementary,** *i.e.* each is a *complement* of the other.

When two straight lines intersect one another, the opposite angles so formed are called **vertically opposite** angles.

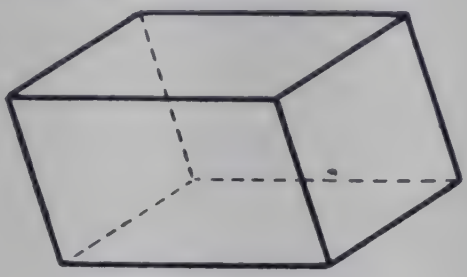
The angles AOB, DOC are **vertically opposite** angles.

SOLIDS.

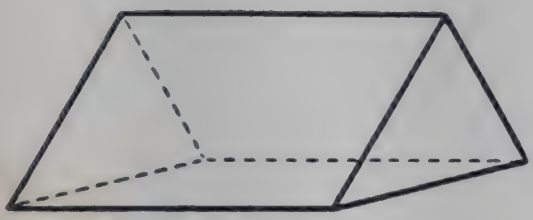
The Forms of Simple Solids.



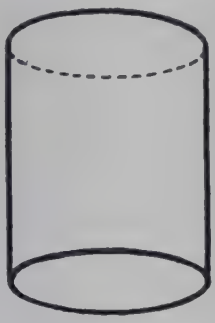
Cube.



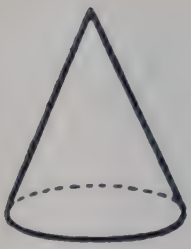
Rectangular Prism.



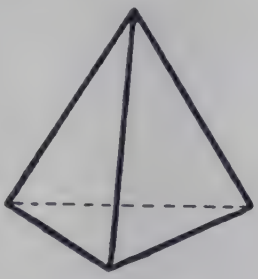
Triangular Prism.



Cylinder.



Cone.

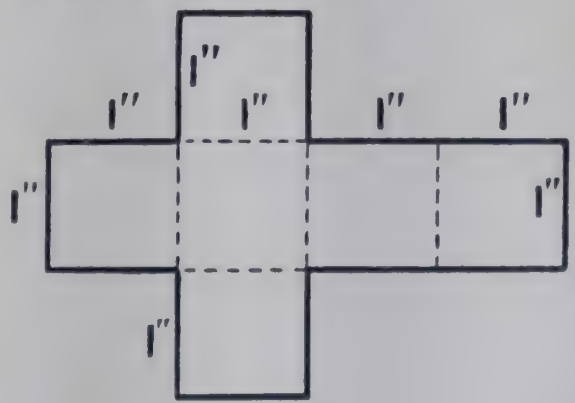


Tetrahedron.

Sphere.

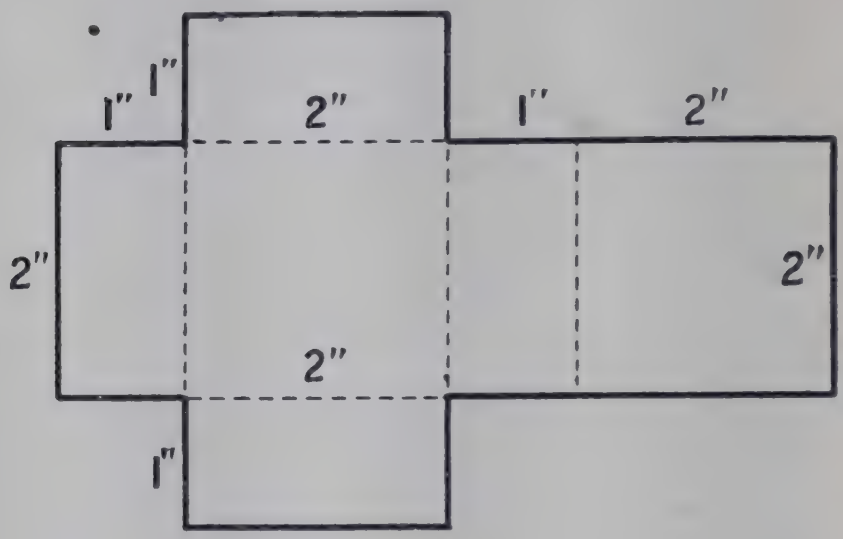
Construction of Models of Solid Figures.

Cube.—On a piece of white paper draw the following figure :—

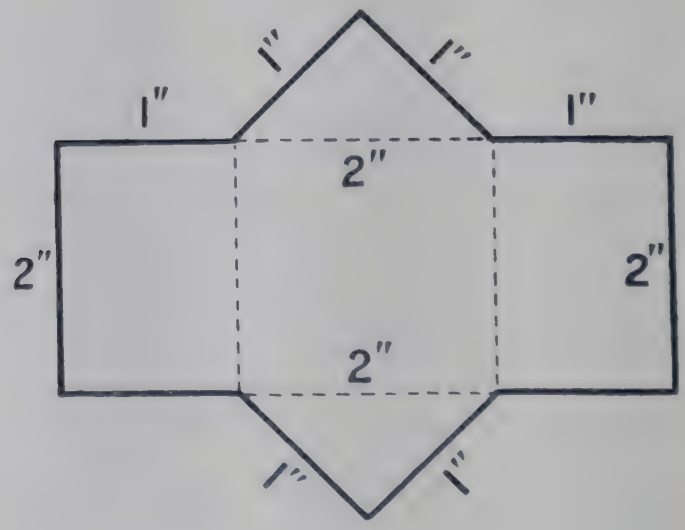


Cut out the figure and fold about the dotted lines. Gum the edges.

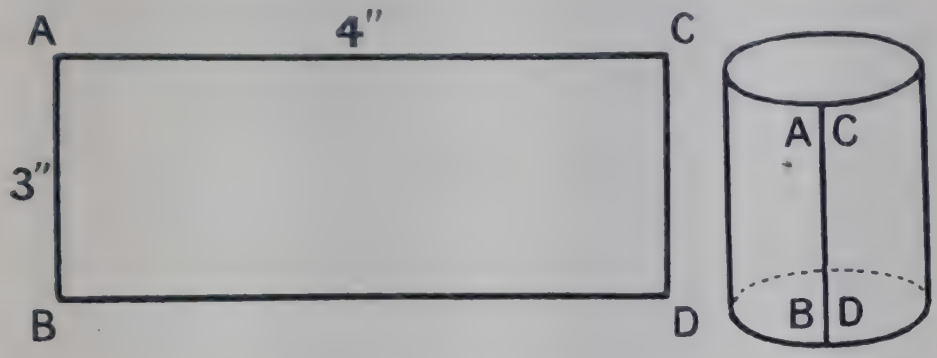
Rectangular Prism.—Construct the following figure, and fold about the dotted lines.



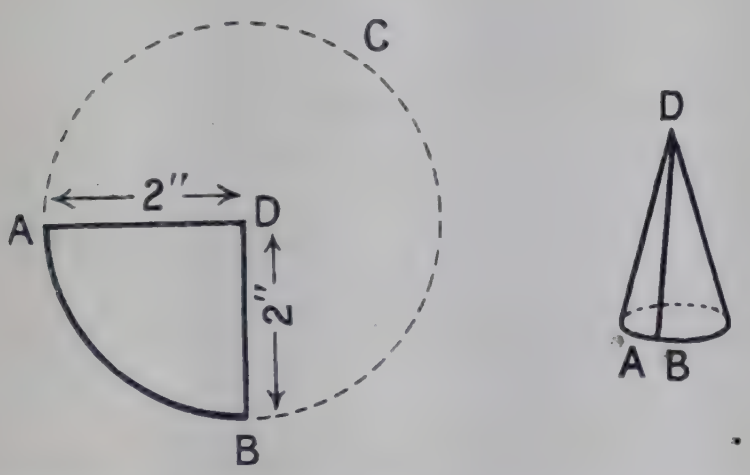
Triangular Prism.—Draw the following diagram. Cut out and fold along the dotted lines.



Cylinder.—Draw a rectangle 3 in. by 4 in. Cut it out and fold to form a cylinder.



Cone.—Draw a circle ABC. Cut out the portion DAB and fold as in diagram.



Tetrahedron.—Construct four equilateral triangles, as shown in figure, and fold about the dotted lines.

The *solids* should be *carefully* examined in respect to—

(*a*) The number of faces.

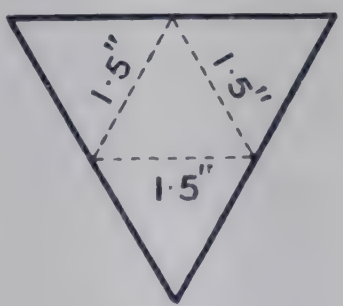
(*b*) The kinds of faces.

(*c*) The opposite faces (where possible).

(*d*) The number of edges.

(*e*) The number of faces meeting at each vertex.

(*f*) The number of vertices, etc.



ANGLES AT A POINT.

Practical Exercises introducing Theorem 1.

(1) Draw any straight line CD standing on another straight line AB ; find the sum of the two adjacent angles so formed.

(2) Draw a similar figure as in question (1) ; measure the adjacent angles ; find their sum. What do you observe ?

(3) Let any two straight lines AB, CD intersect one another at O ; measure all the angles formed, and find their sum. What do you notice ?

(4) Draw any three straight lines AB, CD, EF intersecting one another at O ; measure the angles ; find their sum and write conclusion.

(5) From a point O draw three straight lines OA, OB, OC ; measure the angles so made and find their sum, when no two of the angles overlap.

(6) From a point O draw any number of straight lines you like ; measure the angles contained by them ; find the sum of the angles, when no two of them overlap ; and give your conclusion.

Practical Exercises introducing Theorem 2.

(1) From a point C draw two adjacent angles ACD, DCB equal to two right angles. Are the outside arms AC, CB in the same straight line ? [See Exercise 5, p. 2.]

(2) Repeat the above exercise, taking taking angles 45° and 135°, 60° and 120°. State clearly your conclusion.

(3) From a point O in a straight line AB, straight lines CO, OD are drawn on opposite sides of AB, making angle AOD equal to angle BOC ; prove that COD is a straight line, using tracing-paper.

(4) From a point O in a straight line AB, straight lines CO, OD are drawn at right angles on opposite sides of it ; prove that COD is a straight line.

(5) From a point O in a straight line AB, OC, OD are drawn on opposite sides of AB, so that the angle AOC equals the angle BOD ; prove that COD is a straight line, using tracing-paper.

(6) Draw a figure similar to the one in the diagram with the arms OB, OB, OC, OD, etc., radiating from the point O ; so that angle FOH equals angle EOG ; angle GOC equals angle BOH, and so on ; prove that EOF, BOC, AOB, HOG are straight lines.

Practical Exercises introducing Theorem 3.

(1) Draw two straight lines AB, CD intersecting one another at O and making the angle AOD 85°. Name the vertically opposite angles.

(2) Draw any two straight lines AB, CD cutting one another at O. Measure the vertically opposite angles.

(3) Repeat Exercise 2. What do you observe about the vertically opposite angles?

(4) Two straight lines AOB, COD intersect at O. If the angle DOB is 63°, what are the other angles? Give the answers obtained by (*a*) calculation; (*b*) measurement.

(5) Repeat experiment 4, making the angle DOB 54°.

(6) Make an angle AOB equal to 64°; produce AO to C and BO to D. Measure all the angles. Do they agree with your calculations?

Practical Exercises.

(1) Draw a straight line standing on another straight line, measure the adjacent angles; repeat this exercise three times, placing the lines in different positions. Find the average of the angles formed in each case. What do you infer?

(2) Draw three straight lines AB, AC, AD cutting one another at O. Measure the angles formed and find the average value.

(3) From a point draw any number of straight lines; measure the angles thus formed. What is their sum?

(4) Two straight lines AOB, COD intersect at O; if angle DOB is 75°, what are the values of the other angles?

(5) Make an angle ABC of 45°, and produce the line AB through B to E. What is the value of the angle CBE?

Theoretical Geometry.

In theoretical geometry we have to make use of certain principles which need no demonstration, as they are so self-evident. These self-evident truths are called **axioms**, and serve as a foundation for future reasoning.

Characteristics of an axiom—

(*a*) Self-evident.

(*b*) Incapable of proof or reduction to anything simpler.

(*c*) Fundamental in the conception of space.

An **axiom** may be defined as a self-evident truth which needs no demonstration.

Axioms are of two kinds—

(*a*) **General axioms**, which apply to magnitudes of all kinds.

(*b*) **Geometrical axioms**, which apply only to geometrical magnitudes.

The following are general axioms :—

(1) Things which are equal to the same thing are equal to one another.

(2) If equals be added to equals, the wholes are equal.

(3) If equals be taken from equals, the remainders are equal.

(4) If equals be added to unequals, the wholes are unequal, the greater sum being that which includes the greater of the unequals.

(5) If equals be taken from unequals, the remainders are unequal, the greater remainder being that which is left from the greater of the unequals.

(6) Things which are double of the same thing, or of equal things, are equal to one another.

(7) Things which are halves of the same thing, or of equal things, are equal to one another.

(8) The whole is greater than its part.

Geometrical Axioms.

(9) Magnitudes which can be made to coincide with one another are equal.

(10) Two straight lines cannot enclose a space.

(11) All right angles are equal.

Definitions.

Plane geometry treats of the properties of lines and figures drawn on a plane surface.

Geometry is divided into a number of discussions of relations between parts or determination of parts of a plane figure. The statements of these relations or determinations are termed "Propositions."

There are two kinds of propositions—

(*a*) Problems.

(*b*) Theorems.

A **problem** requires some geometrical construction to be done.

The initial letters Q.E.F. of *Quod Erat Faciendum* (which was to be done) are placed at the end of a problem.

A **Theorem** makes an assertion, the truth of which it proceeds to prove.

The initial letters Q.E.D. of *Quod Erat Demonstrandum* (which was to be demonstrated or proved) are placed at the end of a theorem.

In a theorem—

(1) The **enunciation** is the formal assertion of the theorem.

(2) The **hypothesis** is the relations assumed.

(3) The **conclusion** is the relations proven.

A **corollary** is an inference arising naturally from the discussion of a proposition.

A **postulate** is a single requirement of simple construction necessary for the construction of figures. *E.g.* let it be granted—

(1) That a straight line may be drawn from one point to any other point.

(2) That a terminated straight line may be produced to any length in a straight line.

(3) That a circle may be described with any point as centre, and with any finite straight line as radius.

Abbreviations.

The following abbreviations are allowed of:—

$\therefore$ therefore.	$\perp$ perpendicular to, or right angles to.
$\because$ because.	$>$ is greater than, greater than.
$=$ is, or are, equal to.	$<$ is less than, less than.
$\parallel$ is parallel.	$\odot$ circle.
$\angle$ angle.	$\bigcirc^{ce}$ circumference.
$\triangle$ triangle.	$\square^{m}$ parallelogram.
rt. $\angle$ right angle.	

Other contractions are pt., rect., sq., opp., adj., sim^{ly}., etc., for point, rectangle, square, opposite, adjacent, similarly, etc.

Hints.—(1) Draw bold figures.

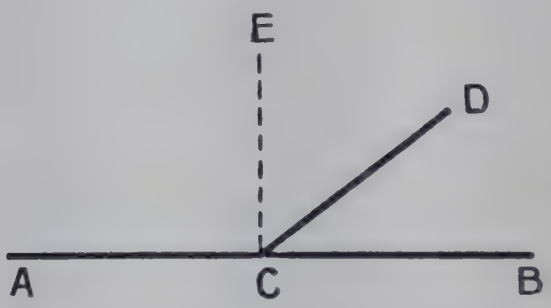
(2) Use Roman letters.

(3) Always give brief statement or reference denoting the property of the proposition used.

Angles at a Point.

Theorem 1.

If a straight line stands on another straight line, the sum of the two adjacent angles so formed is equal to two right angles.



Let the straight line CD meet the straight line AB at C.

Then $\angle ACD + \angle DCB = 2$ rt. $\angle$s.

Construction.—From C draw CE at rt. $\angle$s to AB.

Proof.—$\angle ACD = \angle ACE + \angle ECD$.

To each add $\angle DCB$.

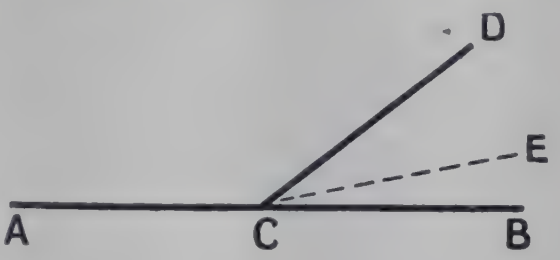
$$\therefore \angle ACD + \angle DCB = \angle ACE + \angle ECD + \angle DCB$$
$$= \angle ACE + \angle ECB$$
$$= 2 \text{ rt. } \angle\text{s (constr.)} \quad \text{Q.E.D.}$$

Corollaries.—(1) If two or more straight lines intersect at the same point, then the sum of the angles so formed is four right angles.

(2) If any number of straight lines radiate from a point, the sum of the angles so formed is four right angles, providing that no two of the angles overlap.

Theorem 2.

If the sum of two adjacent angles is two right angles, their outer arms are in the same straight line.



Let ACD, DCB be two adj. ∠s, and equal to two rt. ∠s.

Then AC, CB are in the same st. line.

Suppose AC, CB be not in the same st. line.

CONSTRUCTION.—Produce AC to E, so that AC and CE be in the same st. line.

PROOF.—∵ ACE is a st. line

∴ ∠ ACD + ∠ DCE = 2 rt. ∠s (I. 1)

and ∠ ACD + ∠ DCB = 2 rt. ∠s (Hyp.),

∴ ∠ ACD + ∠ DCE = ∠ ACD + ∠ DCB,

∴ ∠ DCE = ∠ DCB,

∴ CE coincides with CB,

∴ AC, CB are in the same st. line. Q.E.D.

(Theorem 2 is the converse of Theorem 1.)

A theorem is said to be the **converse** of another when the conclusion of each is the hypothesis of the other.

Exercises on 1 and 2.

(1) If one side of a triangle be produced each way, and the exterior angles be equal to one another, show that the angles at the base of the triangle are equal.

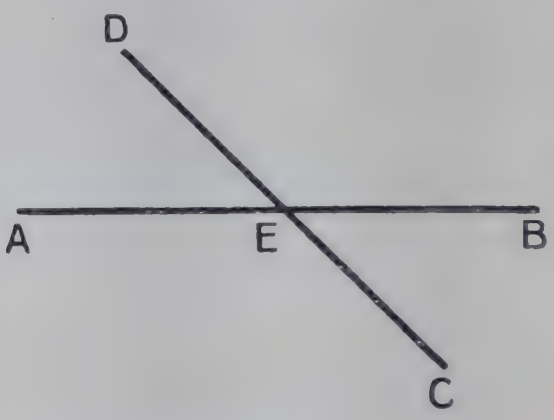
(2) Prove that the bisectors of the adjacent angles which one straight line makes with another contain a right angle.

(3) If two straight lines intersect at a point, and one of the angles so formed is a right angle, prove that the other angles at the point are right angles.

(4) If from a point O in a straight line AB two straight lines OC, OD are drawn at right angles to AB, but on opposite sides of it, prove that DOC is a straight line.

Theorem 3

If two straight lines intersect, the vertically opposite angles are equal.



Let the two st. lines AB and CD cut one another at E.

Then— $\angle$ AED = vert. opp. $\angle$ CEB,

and $\angle$ DEB = vert. opp. $\angle$ AEC.

PROOF.—$\angle$ AED + $\angle$ DEB = 2 rt. $\angle$s (I. 1).

$\angle$ DEB + $\angle$ CEB = 2 rt. $\angle$s.

$\therefore$ $\angle$ AED + $\angle$ DEB = $\angle$ DEB + $\angle$ CEB.

Take away common $\angle$ DEB;

$\therefore$ remg. $\angle$ AED = remg. $\angle$ CEB.

Sim$^{ly.}$ $\angle$ AEC = $\angle$ DEB,

which are vertically opposite angles. Q.E.D.

Exercises.

(1) Let the straight lines AB, CD intersect at O; then, if the angle AOD is bisected by OF, prove that FO produced will bisect the angle BOC.

(2) In the same figure, if the two angles AOD and DOC be bisected, prove that the bisecting lines are in one and the same straight line.

RECTILINEAL FIGURES.

Practical Exercises on Rectilineal Figures.

(1) Draw any closed figure you like having more than four sides (polygon). Measure the lengths of the sides. Give name to polygon.

(2) Draw a circle ABC, having a distance of 2 in. between the legs of the compasses. Starting at any point, divide the circumference into six equal parts by the compasses, keeping the legs of them 2 in. apart. Do ends of compasses come back to initial position? Join the points AB, CD, EF. Measure the length of the lines in centimetres. What kind of a figure is ABCDE?

(3) With a distance of 1·5 in. between the legs of your compasses divide the circumference of a circle ABC. By trial divide the circumference into five equal parts. Join the points as in last exercise. What have you constructed?

(4) Draw a triangle ABC as large as the division of your paper will allow of. Measure the lengths of the sides. Find their sum (the perimeter of the triangle).

(5) Draw a similar figure to 4, and find its perimeter in centimetres.

(6) Draw a regular twelve-sided figure; make use of Exercise 2.

TRIANGLES.

Practical Exercises on Triangles.

(1) Draw a (scalene) triangle having three unequal sides (see Problem 7). Name (*a*) its vertex; (*b*) its base.

(Any side of the triangle may be taken as the base, and then the corresponding vertex is the opposite angular point.)

(2) Describe an equal-sided (equilateral) triangle with compasses only, and test the length of the sides with ruler.

(3) Construct a triangle (isosceles) having two equal sides from the given measurements: AB = 2·5 cm., AC = BC = 4·6 cm.

(4) Draw a triangle ABC having a right angle at B. Mark the right angle. Note the position of the longest side (hypotenuse). Suggest a name for the triangle.

(5) Construct a triangle ABC having an obtuse angle at C. Mark the obtuse angle. Suggest a name for the triangle.

(6) On a base 3·8 cm. long describe a triangle having three acute angles. Give the value of the angles. Suggest a name for the triangle.

DEFINITIONS.

Rectilineal Figures.

Any portion of a plane surface bounded by one or more straight lines is called a **plane figure.**

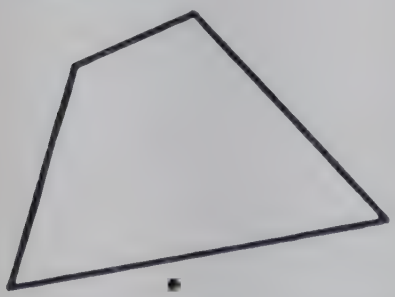
The **perimeter** of a plane figure is the sum of the lengths of its bounding lines.

The **area** of a plane figure is the amount of surface enclosed by its boundary.

A **rectilineal figure** is a plane figure bounded by straight lines.

A **triangle** is a plane figure bounded by three straight lines.

A **quadrilateral** is a plane figure bounded by four straight lines.



A **polygon** is a plane figure bounded by more than four straight lines.

A **convex** polygon is one whose boundary cannot be intersected at more than two points by a straight line drawn across the figure.

(In a convex figure, if the sides be all produced in the same order, then the exterior angles formed are always to the *same* side of the produced sides.)

Equilateral rectilineal figures have all their sides equal.

Equiangular rectilineal figures have all their angles equal.

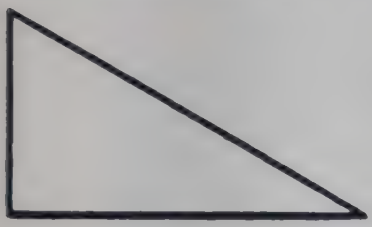
Regular rectilineal figures have all their angles and sides equal.

An **equilateral triangle** is a triangle which has three equal sides.

An **isosceles triangle** is a triangle which has two of its sides equal.

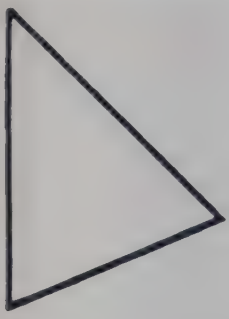
A **scalene triangle** is a triangle which has three unequal sides.

(The curious word "scalene" means uneven or irregular.)



A **right-angled triangle** is a triangle which has a right angle.

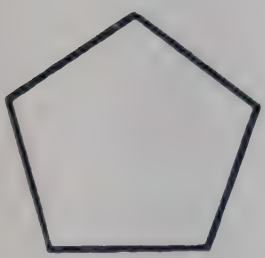
In a right-angled triangle the side opposite the right angle is called the "hypotenuse."



An **obtuse-angled triangle** is a triangle which has an obtuse angle.

An **acute-angled triangle** is a triangle which has three acute angles.

A figure of five sides is called a pentagon.
" six " " hexagon.
" seven " " heptagon.



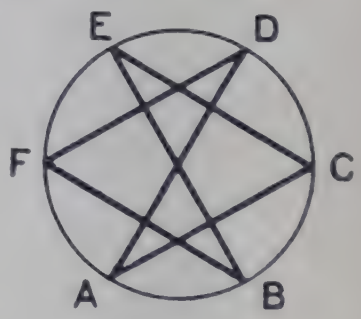
A figure of eight sides is called an octagon.
„ nine „ „ nonagon.
„ ten „ „ decagon.

A **pentagram** is a star-shaped five-sided figure in which each side is met by all the others without the necessity of producing any side.

Construction of a pentagram—

On the circumference of a finely drawn circle take five equidistant points, A, B, C, D, E, and join them in order, AC, CE, EB, BD, DA.

A **hexagram** is a six-sided figure in which any side is met by the opposite side, *i.e.* fourth in order, counting from that side, without the necessity of producing any side.



Construction of a hexagram :—

On the circumference of a finely drawn circle take six equidistant points, A, B, C, D, E, F, and join them thus: AC, CE, EB, BF, FD, DA.

Angles of Rectilineal Figures.

Practical Exercises introducing Theorem 4.

(1) (*a*) Construct a triangle and produce its sides in order.
(*b*) Name the exterior angles.
(*c*) Measure the exterior angles and find their sum.

(2) (*a*) Draw any quadrilateral you like and produce its sides in order.

(*b*) Measure and find the sum of its exterior angles.

(3) (*a*) Draw any five-sided figure and produce its sides in order.

(*b*) Find the sum of the exterior angles.

(4) Find the sum of the exterior angles formed by the sides of a hexagon produced in order.

(5) (*a*) Make a heptagon and produce its sides in order.

(*b*) What is the sum of its exterior angles so formed?

(6) (*a*) Draw on loose paper any two triangles, and produce their sides in order. Cut out the exterior angles so formed, and rearrange the angles of each triangle side by side with a common vertex, and show that the sum of the angles is the same in both cases.

(*b*) Repeat the experiment, taking any other rectilineal figure.

(*c*) Is the sum of the angles the same as in (*a*)?

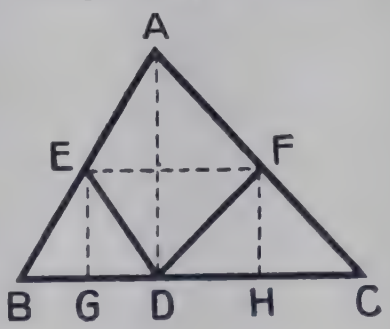
Angles of a Triangle.

Practical Exercises introducing Theorem 5.

(1) Construct a triangle ABC and measure its interior angles; find their sum.

(2) Draw a triangle ABC having AB = 2·4 in., BC = 3·1 in., and CA = 2·8 in. Find the sum of its interior angles.

(3) (*a*) Cut out of paper a large triangle ABC. From D through the apex A draw a perpendicular AD. Fold the triangle about the three dotted lines, making the angles BAC, ACB, and CBA to meet at D.



(*b*) What is the sum of these angles?

(*c*) What do you conclude from the above exercises?

(4) (*a*) Make a right-angled triangle ABC having the angle BAC a right angle.

(*b*) Measure its other angles.

(*c*) What is the sum of the angles CBA and ACB?

(5) Draw any triangle ABC; find the value of each interior angle; add any pair of them together. Compare their sum with two right angles.

(6) Draw an obtuse-angled triangle ABC having the angle ABC an obtuse angle. Find by measurement the value of the angles BAC, ACB ; get their sum. Is the sum greater or less than a right angle?

Practical Exercises on Corollary of Theorem 4.

(1) (*a*) Draw any triangle ABC and measure its angles.
(*b*) What is their sum?

(2) (*a*) Make a quadrilateral, and divide it into two triangles.
(*b*) How many right angles are its interior angles equal to?
(*c*) Verify your answer by measuring the angles.
(*d*) Also verify your answer by drawing the quadrilateral figure on loose paper, and cutting out the angles and rearranging them, side by side, with a common vertex.

(3) (*a*) Construct an irregular pentagon and divide into triangles.
(*b*) Find the sum of its interior angles.

(4) (*a*) Draw a hexagon of any size or shape you wish.
(*b*) By dividing the figure into triangles, determine the sum of its interior angles.

(5) (*a*) Make a heptagon and divide into triangles.
(*b*) Get the value of its interior angles.

(6) (*a*) Tabulate your results as follows :—

Figure.	No. of sides.	Method.
Triangle	3	(3×2) rt. $\angle$s $-$ 4 rt. $\angle$s $=$ 2 rt. $\angle$s.
Quadrilateral	4	(4×2) rt. $\angle$s $-$ 4 rt. $\angle$s $=$ 4 rt. $\angle$s.
Pentagon	5	(5×2) rt. $\angle$s $-$ 4 rt. $\angle$s $=$ 6 rt. $\angle$s.

(*b*) Hence deduce a rule for finding the value of the interior angles of any rectilineal figure.

Exercises.

(1) Find the size of each interior angle of—
(*a*) regular polygon.
(*b*) regular hexagon.
(*c*) regular octagon.
(*d*) regular decagon.

(2) Construct a regular hexagon of 2 in. side.

(3) (*a*) If the two angles of a triangle are 55° and 64°, what is the third angle?

(*b*) Find the magnitudes of the exterior angles formed by producing the sides of the above triangle in order, and their sum.

(4) The sum of the angles at the base of a triangle is 142°; their difference is 40°. Find all the angles.

(5) Three of the angles of a quadrilateral are 54°, 63°, and 75°, what is the fourth angle?

(6) The exterior angle of a regular polygon is 60°; how many sides has the polygon?

Angles of a Triangle.

Practical Exercises introducing Theorem 6.

(1) (*a*) Construct a triangle ABC; produce its sides AB to D.

(*b*) Name the exterior angle, and also the two interior opposite angles.

(2) (*a*) Draw a triangle ABC; produce AB to D.

(*b*) Measure the exterior angle CBD and the interior opposite angles BAC, ACB.

(*c*) Find the sum of the angles BAC, ACB, and compare the value of their sum with the value of the angle CBD.

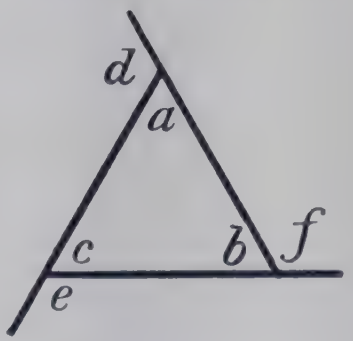
(3) (*a*) Construct a triangle ABC, and produce CB to D.

(*b*) Compare the value of the exterior angle ABD with the value of the sum of the two interior opposite angles BCA, CAB.

(*c*) State what you learn.

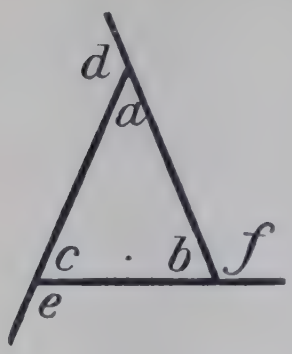
(4) In the figure prove by measurements what interior and opposite angles—

(*a*) exterior angle *f* is equal to.
(*b*) „ „ *d* „ „
(*c*) „ „ *e* „ „



(5) (*a*) In the diagram what angles are equal to the following sums?—

(1) $\angle a + \angle b$.
(2) $\angle b + \angle c$.
(3) $\angle c + \angle a$.



(*b*) What is the value of
$\angle e + \angle f + \angle d$?

(6) (*a*) In the figure what are the sums of?—
(1) $\angle a + \angle d$.
(2) $\angle b + \angle f$.
(3) $\angle c + \angle e$.
(4) $\angle a + \angle b + \angle c$.
(*b*) From your results determine the value of the sum of the exterior angles *d*, *e*, *f*.

Congruent Triangles.

Practical Exercises introducing Theorem 7.

(1) Construct a triangle, having given the length of the sides AB = 2·4 in., AC = 2·8 in., and the included angle BAC = 65° (see Problem 8).

(2) Construct a triangle from the following data : AB = 4·7 cm., AC = 5·8 cm., included angle = 74°.

(3), (4) Draw two triangles having two sides of the one equal to two sides of the other, each to each (5·4 cm. and 4·2 cm.), and the included angle (57°) equal.

(5) (*a*) In space 5 (see Introduction) write down in order as proved by measurement what lines and angles are equal.
(*b*) What do you conclude from your results?

(6) (*a*) Draw two triangles ABC, DEF, given AB = DE = 5·4 cm., AC = DF = 4·7 cm., and the included angle BAC 56°. On a piece of tracing-paper make a trace of triangle DEF. Place the triangle DEF on the triangle ABC, D on A, E on B, and C on F.
(*b*) What do you find?

Angles of a Triangle.

Practical Exercises introducing Theorem 8.

(1) (*a*) Construct an isosceles triangle (see Problem 7).
(*b*) Measure and write down the value of the angles at the base ; compare their value.

(2) Make an isosceles triangle having its equal sides 6·5 cm. long and measure the angles at the base ; compare their value.

(3) (*a*) Draw an isosceles triangle with its equal sides 2·3 in. long, and find the value of the angles at the base.
(*b*) State simply your conclusion from the foregoing exercises.

(4) (*a*) Construct any isosceles triangle and produce the equal sides below the base.
(*b*) Measure the angles on the other side of the base, and compare them.

(5) (*a*) Make an isosceles triangle with equal sides 5·4 cm. long; produce the equal sides below the base; measure the angles formed on the other side of the base.
(*b*) Write down what you learn.

(6) Make an equilateral triangle of 2·8 in. sides, and prove that an equilateral triangle is equiangular.

Exercises.

(1) Construct triangles from the following measurements :—
AB = 5 cm., BC = 8 cm., included ∠ 60°.
AB = 2·4 in., BC = 3·7 in., included ∠ 74°.
AB = 6·8 cm., BC = 4·7 cm., included ∠ 52°.

(2) AB = 4 in., ∠A = 74°, ∠B = 60°.
AB = 6·3 cm., ∠A = 53°, ∠B = 49°.
AB = 9·2 cm., ∠A = 64°, ∠B = 77°.

(3) Construct a triangle with base 3·8 in. long and base angles 53° and 86°.

(4) An upright pole 13 ft. high casts a shadow 18 ft. long. How far from the top of the pole is the extremity of the shadow?

(5) A 60-ft. ladder reaches to a height of 45 ft. against a wall. How far from the wall is the foot of the ladder?

(6) Describe an isosceles triangle having its vertical angle equal to a given angle.

Sides and Angles of a Triangle.

Practical Exercises introducing Theorem 9.

(1) Construct a triangle having the following measurements : ∠ABC = ∠BCA = 60° and base BC = 2·7 in. (see Problem 9), and find the lengths of AB and AC. What kind of a triangle have you made?

(2) Make a triangle given the following data : ∠BAC = ∠ACB = 48° and base AB = 5·6 cm. What are the lengths of AC and BC?

(3) Construct a triangle from the following measurements ; $\angle ACB = \angle CAB = 63°$ and base AC = 6·3 cm. Compare the length of AB with that of BC.

(4) On a plain sheet of paper draw a triangle so that its two base angles are equal. From the vertex drop a perpendicular line to meet the base ; cut out the triangle, and fold the triangle about the perpendicular. What do you learn ?

(5) Repeat the last exercise, constructing an equilateral triangle.

(6) Draw an equiangular triangle and prove that it is equilateral.

Inequalities of Triangles.

Practical Exercises introducing Theorems 10 and 11.

(1) Construct a triangle ABC ; measure the angles at the base. Which side has the greatest angle opposite to it ?

(2) Prove by measurement that if two sides of a triangle are unequal the greater side has the greater angle opposite to it.

(3) Draw a triangle ABC and take AC as the base. Which of the sides BC, BA has the greater angle opposite to it ? What do you conclude from these exercises ?

(4) Construct a triangle ABC having the angle BAC = 64° and the angle BCA = 38°. Which of the angles has the greater side opposite to it ?

(5) Construct a scalene triangle, and prove by measurement that the greater of the base angles has the greater side opposite to it.

(6) Make a triangle ABC, given AB = 3 in., BC = 2·5 in., CA = 2 in. Opposite which angle is the greatest side ? Give conclusion drawn from the last three exercises.

Inequalities of a Triangle.

Practical Exercises introducing Theorem 12.

(1) Construct a triangle ABC. Find the sum of (*a*) AB + BC, (*b*) BC + CA, (*c*) CA + AB.

Compare sum of (*a*) with length of CA.
" " (*b*) " " AB.
" " (*c*) " " BC.

(2) Construct a right-angled triangle ABC, making AB any length you like. Complete the exercise as in the first example.

(3) Construct any isosceles triangle ABC ; find the sum of the lengths of any two of its sides, and compare their sum with the length of the third side.

(4) Repeat Exercise 3, making the base 10 cm., and each of the base angles 5°.

(5) Construct a triangle on the base BC having sides 2·4 in., 1 in., and 1·4 in. Can you do so? If not, why? (Show attempt.)

(6) Construct a triangle with sides 2·8 in., 1·3 in., and 1 in. Show attempt. State precisely what you conclude from the above experiments.

Exercises.

(1) Prove by measurement of a scalene triangle that the difference of any two sides of a triangle is less than the third side.

(2) By construction and measurement show that the sum of the distances of any point from the three angular points of a triangle is greater than half its perimeter.

(3) Prove that the perimeter of a quadrilateral is greater than the sum of its diagonals.

(4) If a point be taken inside a triangle, and three straight lines be drawn to the angles from that point, the sum of these three straight lines shall be less than the perimeter of the triangle.

(5) Any two sides of a triangle are together greater than twice the line drawn from the middle point of the third side to the opposite angle.

(6) ABC is a triangle, and on AB a quadrilateral ADEB is constructed wholly within the triangle. Prove that AC, CB are together greater than AD, DE, and EB of the quadrilateral.

Congruent Triangles.

Figures are said to be **congruent** when they can be made to coincide.

Practical Exercises introducing Theorem 13.

(1) Construct a triangle, having given the base 6·5 cm. and two angles at the base 76° and 54° (see Problem 9).

(2) Construct a triangle from the following measurements: the base 4·5 cm., and the base angles 57° and 69°.

(3), (4) Draw two triangles, having two base angles of the one equal to two base angles of the other (53° and 82°), and also their bases equal (2·2 in.).

(5) In space 5 write down in columns all the lines and angles that you find equal. What is your conclusion from the results?

(6) Draw two triangles, ABC, DEF, having their bases BC, EF each equal to 4·6 cm., and the two angles at the base of the one equal to the two angles at the base of the other (48° and 74°). On tracing-paper make an exact drawing of the triangle DEF, and superimpose it on the triangle ABC. What do you conclude?

Practical Exercises introducing Theorem 14.

(1) Construct a triangle, given the length of the sides AB = 4 cm., BC = 5·8 cm., AC = 5 cm.

(2) Draw a triangle, given AB = 2·4 in., BC = 3·5 in., CA = 1·6 in.

(3), (4) Construct two triangles, ABC, DEF, having three sides of the one equal to the three sides of the other, each to each.

(5) In space 5 write down all the equalities you can find in the two triangles. State clearly your conclusions.

(6) Draw two triangles ABC, DEF having three sides of the one equal to the three sides of the other, each to each. On tracing-paper copy the triangle DEF, and superimpose it on the triangle ABC. What do you observe?

Inequalities of Triangles.

Practical Exercises introducing Theorems 15 and 16.

(1) Construct a triangle ABC, given AB = 5 cm., AC = 6 cm., and the included angle BAC = 65°.

(2) Make two triangles, with two sides of the one equal to two sides of the other, each to each, but the included angles unequal. Compare the remaining side of the one triangle with the remaining side of the other.

(3) Draw two triangles ABC, DEF, having two sides of the one equal to two of the other, but the included angles 64° and 45° respectively. Compare the lengths of their bases. What do you learn?

(4), (5) Make two triangles, having two sides of the one equal to two sides of the other, each to each, but the base of one (6 cm.) greater than the base of the other (4·8 cm.). Measure the included angles BAC, DEF. What do you observe?

(6) Draw two triangles ABC, DEF having two sides of the one equal to two sides of the other, each to each, but the base of one greater than the base of the other. Copy the triangle ABC on tracing-paper, and tell what you observe by superposition of the traced triangle on the triangle DEF.

Congruent Triangles.

Practical Exercises introducing Theorem 17.

(1) Construct an isosceles right-angled triangle ABC, using setsquares.

(2) Draw a right-angled triangle ABC. (See Problem 11.)

(3), (4) Construct two right-angled triangles ABC, DEF having their hypotenuses equal (5·4 cm.), and one side of the one equal to one side of the other (3 cm.).

(5) Find all the equalities you can in (3) and (4), and tabulate in space 5. What conclusion do you arrive at?

(6) Construct two right-angled triangles ABC, DEF having their hypotenuses equal, and one side of the one equal to one side of the other. Make a trace on tracing-paper of the triangle DEF. What do you gather from superposition of the triangle DEF on the other?

Inequalities.

Practical Exercises introducing Theorem 18.

(1) Draw a perpendicular to a straight line AB through a given point C without using setsquares. (See Problem 6.)

(2) Repeat Exercise 1, employing method in Problem 6.

(3) Of all the straight lines that can be drawn to a given straight line from a given point outside it, the perpendicular is the shortest. Prove by measurement this statement.

(4) From a given point C outside a given straight line AB, draw a number of straight lines—including a perpendicular one—to meet the straight line AB. Which is the shortest line?

(5) Repeat Exercise 4. (*a*) Which is the shortest line? (*b*) How do the lengths of the oblique lines vary?

(6) Draw perpendiculars from a point within an equilateral triangle on the three sides, and show that their sum is the same for all positions.

Exercises.

(Draw figures to scale.)

(1) A vertical pole 55 ft. high is found to cast a shadow 40 ft. long. What is the altitude of the sun?

(2) A town A is 8 miles south of a town B, and a town C is 11 miles west of B. How many miles is it from B to C?

(3) A railway track has a gradient of 1 in 20. Show by diagram the angle the track makes with the horizontal, and measure the angle.

(4) A ship is 2·5 miles from a cliff which is 0·5 mile high. Find the angular elevation of the top of the cliff as seen from the ship.

(5) At a point 200 ft. from, and on a level with, the base of a tower, the angle of elevation of the top of the tower is observed to be 60°. What is the height of the tower?

(6) A man 6 ft. high stands a distance of 5 ft. from a lamp-post, and he observes that his shadow is 18 ft. long. Find the height of the lamp-post.

(7) The angle of elevation of a building on a hill from a place due south of it is 60°; and from another place due west of the former, and distant 440 yards from it, is 45°. What is the height of the building?

(8) A 45-ft. ladder reaches to a height of 30 ft. against a wall. How far from the wall is the foot of the ladder?

(9) A ladder leans against a wall. The bottom part of it is 25 ft. from the wall, and the top of it touches the wall 20 ft. from the ground. Find—

(*a*) The length of ladder.
(*b*) Angles with the wall and ground.

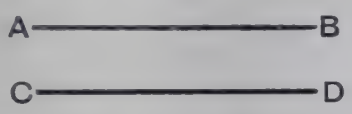
(10) A field is in the form of a triangle; AB and AC are 600 yards and 750 yards long respectively. The angle BAC is 60°. Find the length of BC.

(11) A river is 20 yards wide. Two men (A and B) stand on opposite banks, exactly opposite to each other. Another man C is 12 yards from A on the same side of the river. How far is C from B?

(12) A pole is 110 ft. high. It casts a shadow 80 ft. long. What is the altitude of the sun? What is the distance between the foot of the pole and the end of the shadow?

Direction and Sense.

A straight line is generated, as already proved, by the motion of a point in any fixed direction. The point may move in either of two opposite directions or *senses*—from right to left or from left to right. Now, a number of parallel straight lines may be drawn all in the same sense, or some in one sense and some in another, preserving the same direction. *E.g.* take two parallel straight lines **AB**, **CD**.



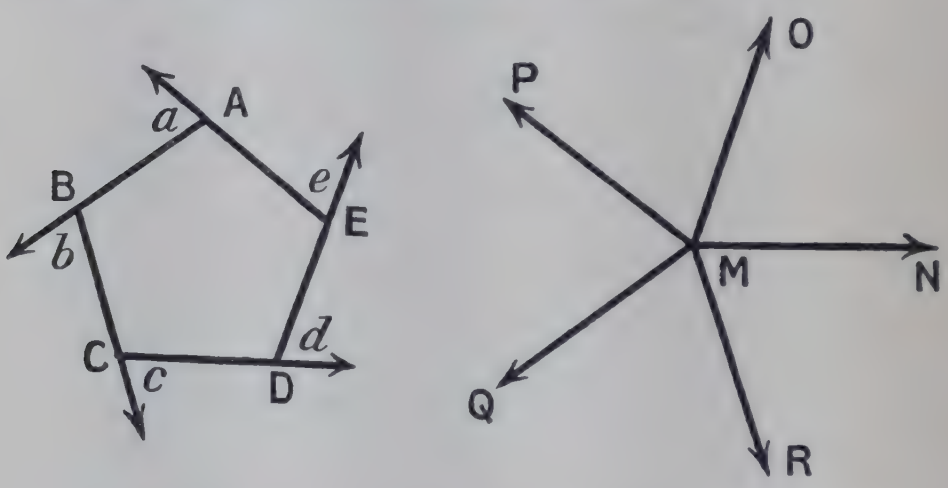
If a point moves from **A** to **B**, and another from **C** to **D**, we say that they move *in the same sense;* but if the one moves from **A** to **B**, and the other from **D** to **C**, they move in opposite senses. **AB** and **DC** we say have *opposite senses*.

An **angle** between two straight lines is necessarily equal to two parallel lines in the same sense.

Angles of a Polygon.

Theorem 4.[1]

If the sides of a convex polygon are produced in order, the sum of the angles so formed is equal to four right angles.



Let ABCDE be a convex polygon, having its sides produced in order, forming the ext. $\angle$s a, b, c, d, e.

∴ Then sum of $\angle$s a, b, c, d, e = 4 rt. $\angle$s.

Construction.—Through any pt. M draw MP, MQ, MR, MN, MO ‖ to and in the same *sense* as EA, AB, BC, CD, DE respectively.

Proof.—∵ MP and MQ are respectively ‖ to and in the same sense as EA, AB,

$$\therefore \angle a = \angle \text{PMQ}.$$

Sim$^{\text{ly}}$. $\angle b = \angle$ QMR,

$\angle c = \angle$ RMN,

$\angle d = \angle$ NMO,

$\angle e = \angle$ OMP.

$\therefore \angle a + \angle b + \angle c + \angle d + \angle e =$ sum of $\angle$s at M

= 4 rt. $\angle$s (I. 1, Cor.).

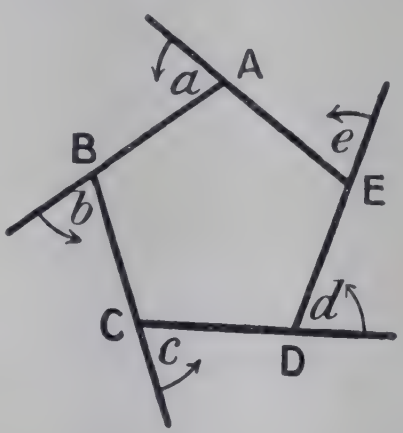
[1] This proof should be omitted at the first reading, the alternative proof to be taken instead.

Theorem 4 (Alternative Proof).

If the sides of a convex polygon are produced in order, the sum of the angles so formed is equal to four right angles.

Let ABCDE be any convex polygon, having its sides produced in order, forming the ext. $\angle$s a, b, c, d, e.

Then sum of $\angle$s a, b, c, d, e = 4 rt. $\angle$s.



Proof.—Suppose a st. line to be on CD, and to rotate about the pt. D through the $\angle$ d; it will then be on DE. Suppose the line to slide along DE till the pt. previously at D lies on E; imagine the line to rotate about E through the $\angle$ e. Further, suppose the line to slide along the line EA, and to turn about A through the $\angle$ a. Again, imagine the line to slide along AB, and to rotate about B through the $\angle$ b. Similarly, the line to slide along BC, and to turn about C through the $\angle$ c.

The rotated line will now be in the same position and have the same direction as it had at first. Therefore the total rotation is equal to one complete turn, or 4 rt. $\angle$s.

But the line rotated through the ext. $\angle$s a, b, c, d, e,

$\therefore$ the sum of these $\angle$s = 4 rt. $\angle$s. Q.E.D.

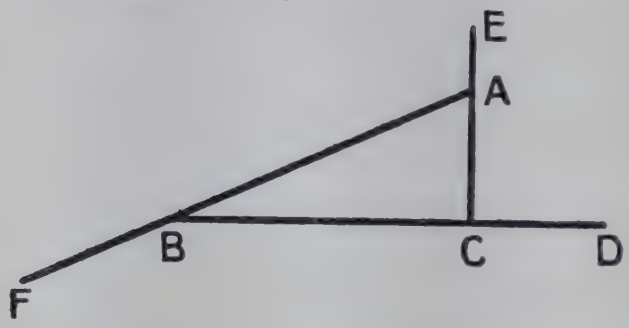
Corollary.—If the sides of any convex rectilineal figure be produced in order, the sum of the angles so formed is equal to four right angles.

Angles of a Triangle.

Theorem 5.

The sum of the interior angles of any triangle is equal to two right angles.

Let **ABC** be any △.



Then sum of its ∠s **ABC, BCA, CAB** = 2 rt. ∠s.

Construction. —Produce the sides the same way round to **D, E, F.**

Proof. — The sum of ∠s **ABC, CBF** = 2 rt. ∠s (I. 1).

Sim$^{ly.}$ the sum of ∠s **BCA, ACD** = 2 rt. ∠s,

and the sum of ∠s **CAB, EAB** = 2 rt. ∠s,

the sum of the int. ∠s + ext. ∠s = 6 rt. ∠s.

But the sum of the ext. ∠s = 4 rt. ∠s (I. 4, Cor.);

∴ the sum of the int. ∠s **ABC, BCA, CAB** = 2 rt. ∠s.

Q.E.D.

Corollaries.—(1) Any two angles of a triangle are together less than two right angles.

(2) In any triangle, one only of its angles can be a right or obtuse angle.

(3) In any triangle there must be at least two acute angles; there may be three.

(4) The sums of the interior angles of different triangles are always equal.

(5) If the sum of two angles of a triangle be equal to the sum of two angles of another triangle, the remaining angles are equal.

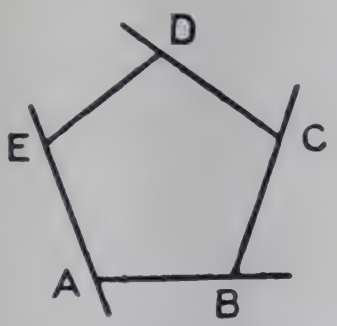
(6) If one of the angles of a triangle be a right angle or an obtuse angle, the remaining angles must be acute.

(7) If a triangle contains a right angle, then the other angles are complementary.

ANGLES OF A POLYGON.

Corollary of Theorem 4.

All the interior angles of any convex rectilineal figure, together with four right angles are equal to twice as many right angles as the figure has sides.



Let **ABCDE** be a rectilineal figure of n sides.

Then all its int. $\angle$s + 4 rt. $\angle$s = $2n$ rt. $\angle$s.

CONSTRUCTION.—Produce the sides of the polygon in order.

PROOF.—Each int. $\angle$ + corres^{g}. ext. $\angle$ = 2 rt. $\angle$s. (I. 1);

$\therefore$ all int. $\angle$s + corres^{g}. ext. $\angle$s = twice as many rt. $\angle$s as the figure has sides.

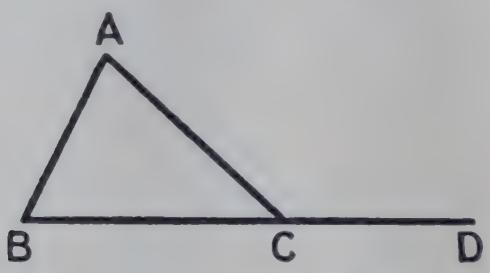
But all ext. $\angle$s = 4 rt. $\angle$s (I. 5);

$\therefore$ all int. $\angle$s + 4 rt. $\angle$s = twice as many rt. $\angle$s as the figure has sides. Q.E.D.

Angles of a Triangle.

Theorem 6.

If one side of a triangle be produced, the exterior angle is equal to the sum of the two interior opposite angles.



Let **ABC** be any △ having its side **BC** produced to **D**.

Then ext. ∠ **ACD** = int. opp. ∠s **CBA, BAC**.

Proof.—Sum of ∠s **ACD, ACB** = 2 rt. ∠s (I. 1),

Sum of int. ∠s **ACB, CBA, BAC** = 2 rt. ∠s (I. 5),

From these equals take away the common ∠ **ACB**,

∴ rem^{r} ext. ∠ **ACD** = int. opp. ∠s **CBA, BAC**. Q.E.D.

Corollary.—If one side of a triangle be produced, the exterior angle is greater than either of the interior opposite angles.

Exercises.

(1) If the alternate sides of a hexagon be produced to meet, the angles at the points of intersection will be equal to four right angles.

(2) How many sides has the rectilineal figure, the sum of whose interior angles is double the sum of its exterior angles?

(3) If the sides of a pentagon be produced to meet, the angles at the points of intersection will be equal to two right angles.

(4) Trisect a right angle.

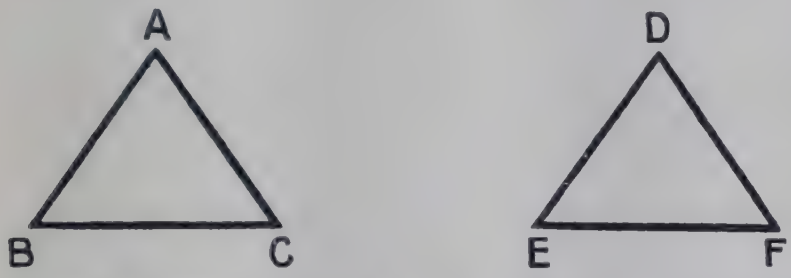
(5) Name the magnitude of an angle of a regular octagon.

(6) Any two exterior angles of a triangle are together greater than two right angles.

Congruent Triangles.

Theorem 7.

If two triangles have two sides of the one equal to the two sides of the other, each to each, and the included angle of the one equal to the included angle of the other, then the two triangles are congruent.



Let ABC, DEF be two △s with side AB = side DE, side AC = side DF and incl. ∠ BAC = incl. ∠ EDF.

Then base BC = base EF,

∠ ABC = ∠ DEF,

∠ ACB = ∠ DFE.

Area of △ ABC = area of △ DEF.

Proof.—Let △ ABC be placed upon △ DEF so that pt. A is on pt. D and st. line AB lies along st. line DE; then pt. B will fall on pt. E.

∵ AB = DE (Hyp.),

If AB lies on DE, then AC lies along DF.

∵ ∠ BAC = ∠ EDF (Hyp.),

and ∵ AC = DF, the pt. C falls on pt. F.

∴ BC will also coincide with EF.

∴ △ ABC coincides with △ DEF,

∴ △s ABC, DEF are congruent. Q.E.D.

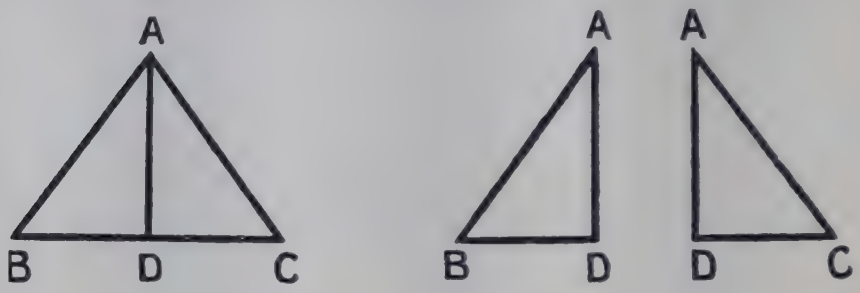
Exercises.

(1) Prove that the diagonals of a square are equal.

(2) Prove that the squares described on two equal straight lines are equal.

Theorem 8.

The angles at the base of an isosceles triangle are equal.



Let ABC be an isosceles △ with side AB = side AC.

Then ∠ ABC = ∠ ACB.

Construction.—Bisect ∠ BAC by the st. line AD, which meets the base at D.

Proof.—In △s ABD, ACD

AB = AC,

AD is common,

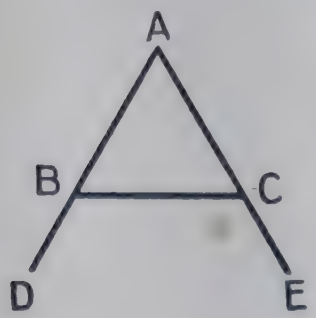
incl. ∠ BAD = incl. ∠ CAD.

∴ △s ABD, ACD are congruent (I. 7);

∴ ∠ ABD = ∠ ACD,

i.e. ∠ ABC = ∠ ACB. Q.E.D.

Corollary.—(1) The angles on the other side of the base of an isosceles triangle are equal.



Construction.—Produce sides AB, AC to D and E respectively.

Then ∠ CBD = ∠ BCE.

Proof.— ∵ CB meets st. line AD at B,

∠s ABC, CBD = 2 rt. ∠s (I. 1).

Similarly ∠s ACB, BCE = 2 rt. ∠s.

∴ ∠s ABC, CBD = ∠s ACB, BCE,

∵ ∠ ABC = ∠ ACB;

∴ rem$^{s.}$ ∠ CBD = rem$^{s.}$ ∠ BCE.

COROLLARY.—(2) Every equilateral triangle is also equiangular.

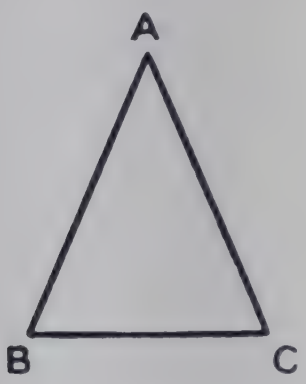
Exercises.

(1) Prove that the opposite angles of an equilateral four-sided figure are equal.

(2) If the angles formed by the base of a triangle and the sides produced be equal, the sides of the triangle are equal.

Theorem 9 (Converse of Theorem 8).

If two angles of a triangle are equal, the sides opposite to these angles are equal.



Let ABC be a △ with ∠ ABC = ∠ ACB.

Then side AB = side AC.

PROOF.—Suppose the △ ABC be lifted up, inverted, and laid down on its trace, so that C falls on B, and B on C.

Then CA falls along BA, and BA along CA,

∵ ∠ ABC = ∠ ACB (Hyp.),

CA falling on BA, and BA on A,

∴ A falls back on its original position,

∴ CA coincides with AB;

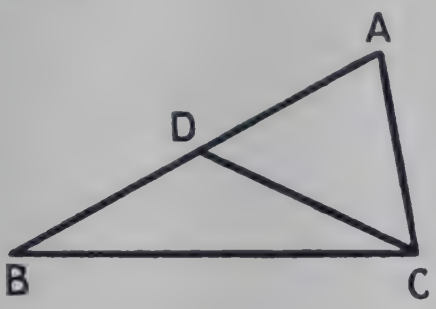
∴ AB = AC. Q.E.D.

COROLLARY.—An equiangular triangle is also equilateral.

INEQUALITIES.

Theorem 10.

If two sides of a triangle are unequal, the greater side has the greater angle opposite to it.



Let ABC be a △ in which AB > AC.

Then ∠ ACB > ∠ ABC.

CONSTRUCTION.—From AB cut off AD = AC. Join DC.

PROOF.—∵ △ ADC is isosceles and DC the base,

∴ ∠ ADC = ∠ ACD (I. 8).

But ∵ the side BD of △ BDC is produced to A;

∴ ext. ∠ ADC > int. opp. ∠ ABC (I. 6, Cor.);

∴ ∠ ACD > ∠ ABC.

But ∠ ACB > its part ∠ ACD;

∴ ∠ ACB > ∠ ABC. Q.E.D.

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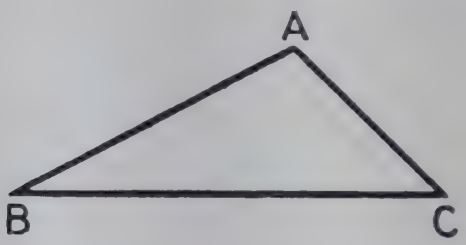
Theorem 11 (Converse of Theorem 10).

If any two angles of a triangle are unequal, the greater angle has the greater side opposite to it.



Let ABC be a $\triangle$ in which $\angle$ ACB $>$ $\angle$ ABC.

Then side AB $>$ side AC.

PROOF.— If AB $=$ AC,

then $\angle$ ACB $=$ $\angle$ ABC (I. 8);

but this is not so.

If AB $<$ AC,

then $\angle$ ACB $<$ ABC (I. 10);

but this is not so,

$\therefore$ AB must be $>$ AC. Q.E.D.

Exercises.

(1) Let AB be drawn at right angles to CD, and meet it in B; join AC, AD. If BC is greater than BD, show that AC is greater than AD.

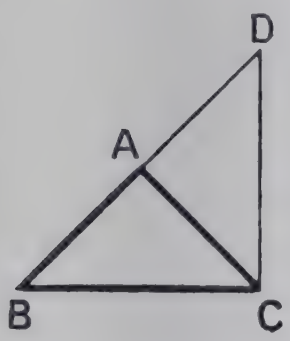
(2) ABCD is a quadrilateral in which the side AD equals the diagonal AC; show that the side BC is shorter than the diagonal BD.

(3) Prove that the hypotenuse is the greatest side of a right-angled triangle.

(4) If within a triangle ABC there be such a point D that AC and AD are equal, show that AB is longer than AC.

Theorem 12.

Any two sides of a triangle are together greater than the third side.



Let ABC be any △.

Then any two of its sides are together > the third side.

$$BA + AC > BC.$$
$$AC + CB > BA.$$
$$CB + BA > CA.$$

CONSTRUCTION.—Produce BA to D, making AD = AC. Join DC.

PROOF.— In △ ACD, AC = AD (Const.);

$\therefore \angle ACD = \angle ADC$ (I. 8).

Now $\angle BCD >$ its part $\angle ACD$;

$\therefore \angle BCD$ is also $> \angle ADC$;

$\therefore$ in △ DBC $\angle BCD > \angle BDC$;

$\therefore BD > BC$;

i.e. $BA + AD > BC$;

$\therefore BA + AC > BC$ (Const.).

Sim$^{ly.}$ $AC + CB > BA$,

and $CB + BA > CA$. Q.E.D.

COROLLARY.—The difference of any two sides of a triangle is less than the third side.

Exercises.

(1) In the case of any rectilineal figure, the sum of all the sides, except one, is greater than the excepted one.

(2) If a point be taken inside a triangle, and three straight lines be drawn to the angles from that point, the sum of these three straight lines shall be less than the perimeter of the triangle.

(3) The sum of the distances of any point from the three angles of a triangle is greater than half the perimeter of a triangle.

(4) Prove that the straight line is the shortest distance between two points by the help of Exercise 1.

(5) Discuss Exercise 4 to show that a curved line joining two points is greater than the straight line joining them.

(6) If two polygons stand on the same base, and one falls inside the other, then the perimeter of the outside one is always greater than the inside one, provided the inside one is convex.

(7) If one polygon falls entirely within another one, then the perimeter of the exterior one is always greater than the perimeter of the interior one, provided the inside one is convex.

(8) Prove by drawing a regular inscribed hexagon that the circumference of a circle is greater than three times its diameter.

N.B.—In working out Examples 6 and 7, produce the sides of the interior polygon in the same order, and apply Exercise 5.

CONGRUENT TRIANGLES.

Theorem 13.

If two triangles have two angles of the one equal to two angles of the other, each to each, and also one side of the one equal to the corresponding side of the other, the triangles are congruent.

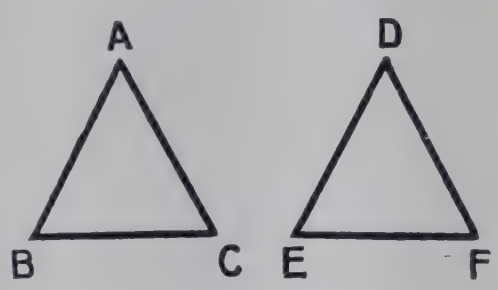
Let △s ABC, DEF have BC = EF, and ∠ ABC = ∠ DEF and ∠ ACB = ∠ DFE.

Then the △s ABC, DEF are congruent.

PROOF.—∵ two ∠s of △ ABC respectively = two ∠s of △ DEF,

∴ rem$^{g.}$ ∠ BAC = rem$^{g.}$ ∠ EDF (I. 6, Cor.).

Apply the △ ABC to the △ DEF, so that B falls on E and BC coincides with EF.



Then ∵ BC = EF,

C falls on F;

∵ ∠ ABC = ∠ DEF,

Then BA must fall along ED.

Again, ∵ ∠ ACB = ∠ DFE,

Then CA must fall along FD;

∴ A, which falls both on ED, FD, must coincide with D;

∴ △ ABC coincides with △ DEF;

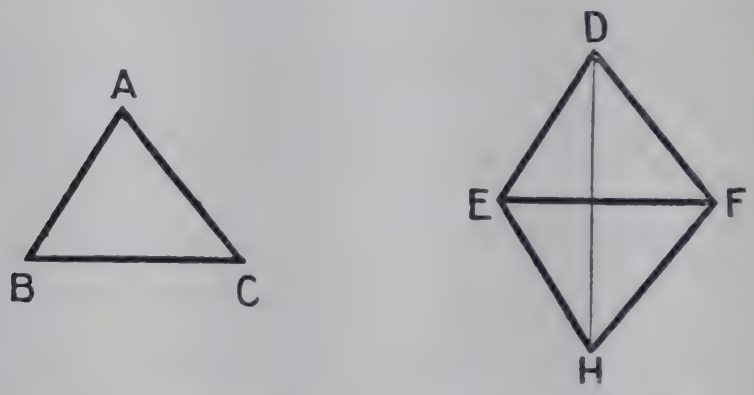
∴ △s ABC, DEF are congruent. Q.E.D.

Exercise.

(1) Show that the perpendicular drawn from the extremities of the base of an isosceles triangle to the opposite sides are equal.

Theorem 14.

If two triangles have three sides of the one equal to the three sides of the other, each to each, the triangles are congruent.



Let ABC, DEF be two △s in which AB = DE, BC = EF, CA = FD.

Then the △s are congruent.

Proof.—Apply the △ ABC so that B falls on E, and BC coincides with EF, and A and D are on opp. sides of EF. Let H be the new position of A. Join DH.

In △ EDH, ED = EH (Hyp.),

∴ ∠ EDH = ∠ EHD (I. 8).

In △ DFH, FD = FH (Hyp.),

∴ ∠ FDH = ∠ FHD (I. 8),

∴ ∠ EDH + ∠ FDH = ∠ EHD + ∠ FHD;

∴ ∠ EDF = ∠ EHF;

i.e. ∠ EDF = ∠ BAC.

Now, in △s ABC, DEF

AB = BE,

AC = DF,

and incl. ∠ BAC = incl. ∠ EDF,

∴ △ ABC = △ DEF (I. 7);

∴ the △s are congruent. Q.E.D.

INEQUALITIES.

Theorem 15.

If two triangles have two sides of the one equal to two sides of the other, each to each, and the included angles unequal, then the base of that with the greater included angle is greater than the base of the other.

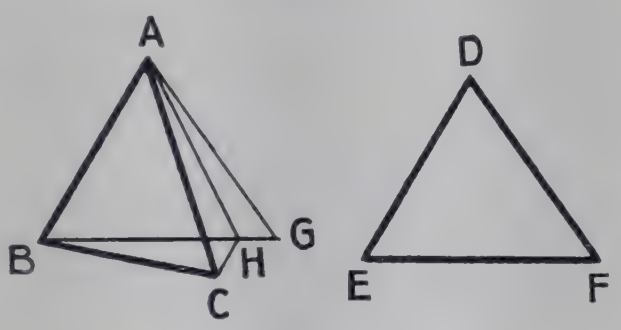
Let ABC, DEF be two △s which have AB = DE, AC = DF, but incl. ∠ EDF > incl. ∠ ABC.

Then base EF > base BC.

PROOF. — Apply △ EDF to △ ABC so that D falls on A and DE falls along AB; then E falls on B (Hyp.).

∵ ∠ EDF > ∠ BAC,

∴ DF falls outside ∠ BAC.



Let △ EDF take up the position ABG.

Bisect ∠ CAG by the st. line AH, meeting BG in H. Join HC.

In △s CAH, GAH

∵ AC = AG,

AH is common,

and incl. ∠ CAH = incl. ∠ GAH (Const.),

∴ △s CAH, GAH are congruent (I. 7).

And in △ BHC

BH + HC > BC (I. 12).

But HC = HG (proved),

∴ BH + HG > BC;

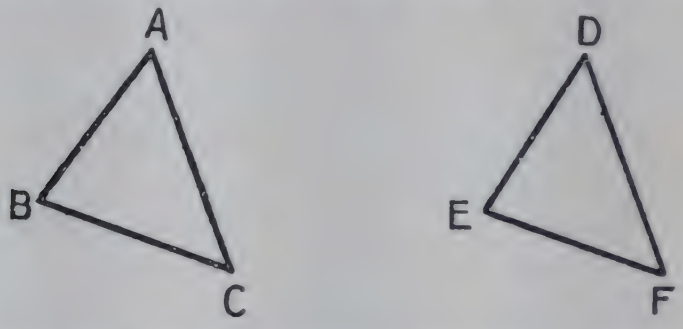
i.e. BG > BC;

i.e. EF > BC. Q.E.D.

(*This proposition may be omitted.*)

Theorem 16 (Converse of Theorem 15).

If two triangles have two sides of the one equal to two sides of the other, each to each, and their bases unequal, then the angle contained by the sides of that which has the greater base is greater than the angle contained by the corresponding sides of the other.



Let ABC, DEF be two △s which have AB = DE, AC = DF, and BC > EF.

Then ∠ BAC > ∠ EDF.

PROOF.—For if ∠ BAC be not > ∠ EDF,

then either ∠ BAC must be = or < ∠ EDF.

If ∠ BAC = ∠ EDF,

then BC = EF (I. 7);

but it is not.

If ∠ BAC < ∠ EDF,

then BC < EF (I. 11);

but it is not;

∴ ∠ BAC is neither equal to nor less than ∠ EDF;

∴ ∠ BAC > EDF. Q.E.D.

(*This proposition may be omitted.*)

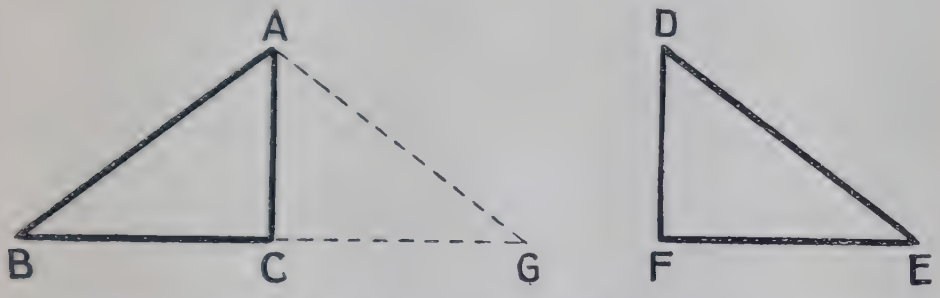
CONGRUENT TRIANGLES.

Theorem 17.

If two right-angled triangles have their hypotenuses equal, and one side of the one equal to one side of the other, the triangles are congruent.

Let ABC, DEF be two △s which have their hypotenuses AB, DE equal, and AC = DF and ∠s ACB, DFE rt. ∠s.

Then the △s ABC, DEF are congruent.



PROOF.—Apply the △ DEF so that D falls on A and DF along AC and the △ DEF takes up the position AGC.

∵ DF = AC,

∴ F falls on C;

∵ ∠s ACB, ACG are rt. ∠s,

∴ BCG is a st. line (I. 2).

Now in △ ABG ∵ AB = AG (Hyp.),

∴ ∠ ABG = ∠ AGB (I. 8).

Again in △s ABC, AGC

∠ ABC = ∠ AGC,

∠ ACB = ∠ ACG,

AB = AG (Hyp.),

∴ △s ABC, AGC are congruent (I. 13),

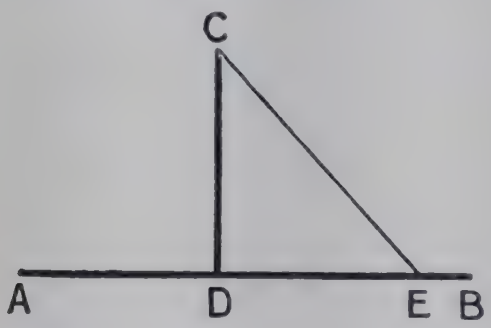
i.e. △s ABC, DEF are congruent. Q.E.D.

INEQUALITIES.

Theorem 18.

Of all the straight lines that can be drawn to a given straight line from a given point outside it the perpendicular is the shortest.

Let AB be a strt. line and C a pt. outside it, and DC be drawn ⊥ to AB, meeting it at D.



Then CD is the shortest line that can be drawn from C to AB.

CONSTRUCTION. — Draw any other strt. line from C to meet AB at E.

PROOF.—∵ the side DE of △ CDE is produced to A,

∴ ext. ∠ CDA > ∠ CED (I. 6, Cor.),
and ∠ CDA = ∠ CDE (Hyp.);
∴ ∠ CDE > ∠ CED;
∴ CE > CD (I. 11).

Sim^ly. CD may be proved shorter than any other straight line drawn from C to meet AB,

∴ CD is the shortest of all lines that can be drawn from C to meet AB. Q.E.D.

Miscellaneous Exercises.

(1) Name the conditions under which triangles are congruent.

(2) Give the properties of a triangle relating to--
(*a*) The sum of its exterior angles.
(*b*) The sum of its interior angles.

(3) If D is any point in the side AC of a triangle ABC, prove that BA + AC > BD + DC.

(4) If the bisector of the vertical angle of a triangle is at right angles to the base, the triangle is isosceles.

(5) If one side of a regular hexagon be produced, show that the exterior angle is equal to the interior angle of an equilateral triangle.

(6) Express in degrees the magnitude of each exterior angle of a regular heptagon.

(7) The straight line joining the middle point of the hypotenuse of a right-angled triangle to the right angle is equal to half the hypotenuse.

(8) ABC is an isosceles triangle. A straight line is drawn at right angles to the base, and cuts the side or sides produced in D and E. Prove that triangle ADE is isosceles.

(9) ABC is an isosceles triangle (AC = BC); the bisectors of the $\angle C$ and $\angle B$ meet at D. Prove that DA bisects $\angle A$.

(10) In two triangles ABC, DEF $\angle A = \angle D$, $\angle B = \angle E$; prove that $\angle C = \angle F$.

(11) ABCD is a square, E is the middle point of AB; join CE, DE. Prove that the triangle AED equals the triangle BEC.

(12) Show that the triangle formed by joining the middle points of the sides of an equilateral triangle is also equilateral.

(13) ABC is a triangle having the sides AB, AC equal. In AB take any point D; produce CA to E, making AE equal to AD. Show that ED produced cuts BC at right angles.

(14) ABC is a right-angled isosceles triangle, of which AB is the hypotenuse; AD is drawn bisecting the angle BAC, and meeting BC in D. Show that AC + CD = AB.

(15) Show that the perpendiculars drawn from the extremities of the base of an isosceles triangle to the opposite sides are equal.

(16) In any triangle ABC the bisectors of the angles ABC, ACB meet in O. Show that OA bisects the angle BAC.

(17) If the bisector of an angle of a triangle cuts the opposite side at right angles, the triangle must be isosceles.

(18) The perpendiculars from the vertices to the opposite sides of an equilateral triangle are equal to one another.

(19) The extremities of a given straight line are equidistant from any straight line drawn through its middle point.

(20) The bisector of the $\angle BAC$ of a triangle ABC meets BC in D, and BC is produced to E. Prove that the angles ABC, ACE equal twice angle ADC.

(21) If two straight lines bisect each other at right angles, any point in either of them is equidistant from the extremities of the other.

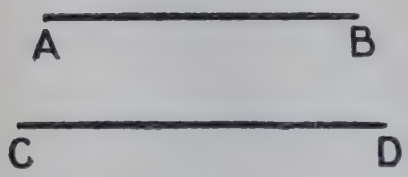
(22) If the angles at the base of an isosceles triangle ABC be bisected by the straight lines BD and CD, prove that DBC is an isosceles triangle.

(23) Find a point in a given straight line such that its distance from two given points may be equal.

(24) If three points be taken on the sides of an equilateral triangle at equal distances from the angles and the points joined, then the triangle so formed will also be equilateral.

(25) The sum of the distances from any point from the three angles of a triangle is greater than half the sum of the sides of the triangle.

Parallels.

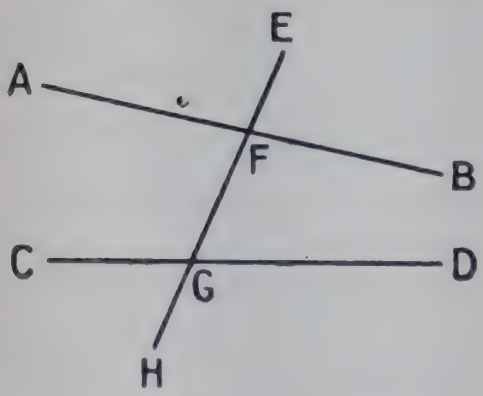


Parallel straight lines are such as, being in the same plane, do not meet, however far they are produced in either direction.

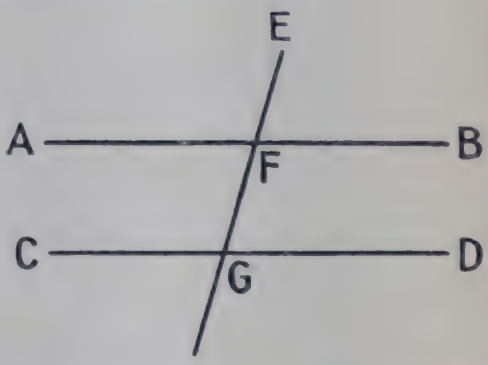
PROPERTIES.—(*a*) In the same plane.

(*b*) Being produced, they will never meet in either direction.

In the figure two straight lines, AB, CD, are cut by a third straight line EH at F and G. For the purpose of distinguishing the angles so formed, particular names are given.



Ex. (*a*) ∠ AFG and ∠ FGD are **alternate** angles.



(*b*) ∠ EFB and ∠ EGD are **corresponding** angles.

If we have CD parallel to AB, we may regard CD as AB moved into a new position without altering its direction. Then angle EGD is angle EFB in its new position. Hence we call them **corresponding** angles.

The angles **EFB, EGD** are sometimes referred to as an **exterior** angle and the **interior opposite** *angle* on the same side of the intersecting line.

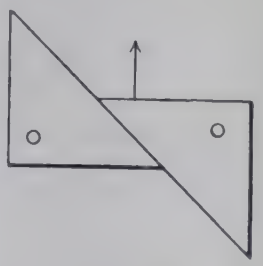
In Theorem 21 we have to assume some further fact about parallels. This further fact is stated by Euclid in his Twelfth Axiom. It does not appear obvious. All attempts of mathematicians at proving this axiom have proved futile. It is a truth which lies at the foundation of **Euclidean** geometry. However, by denying this axiom a consistent geometry can be written, and such geometry is known as **Non-Euclidean.** In Non-Euclidean geometry, generally, it is assumed that through a given point two different straight lines can be drawn parallel to a given straight line on either side; whereas in Euclidean geometry, through a given point, only one parallel can be drawn to a given straight line. Thus—

Two intersecting straight lines cannot both be parallel to a third straight line.

This is known as **Playfair's Axiom**, and is equivalent to the twelfth axiom of Euclid.

Practical Exercises introducing Theorem 19.

(1) Draw a straight line parallel to a given straight line, using set squares.



(2) Draw a straight line parallel to a given straight line through given point outside it (see Problem 12).

(3) Draw a straight line parallel to a given straight line (see Problem 13).

(4) Draw a straight line parallel to any given straight line; cut them by another straight line; indicate and name all the pairs of alternate angles.

(5) On a straight line AB make an angle BAE 60°; on AE produced make another angle AED 60° on the opposite side. What kind of angles are DEA, AEB? Test by set square if DE and AB are parallel. What do you learn?

(6) Repeat Exercise 5, but taking a right angle. Tell what you learn from the exercise.

Practical Exercises introducing Theorem 20.

(1) Let a straight line cut two parallel straight lines ; indicate in the figure the corresponding angles, and under the diagram write down the names of the corresponding angles.

(2) Draw a straight line cutting two parallel straight lines ; name and indicate the interior angles on the same side of the line.

(3) In the straight line AB take two points C, D. At C make the angle DCE 45° ; at D make the angle ADF 45° on the opposite side. By means of setsquares see if line CE be parallel to DF. Give answer and conclusion.

(4) Repeat the last exercise, but taking an angle of 120°. Tell what you learn.

(5) Draw a straight line AB ; on it take any two points C, D. At C make any angle you like ; at D make its supplement, so that the two angles may be facing each other. Test if CE be parallel to DF. Give answer and conclusion.

(6) Repeat Exercise 5, using angles of 45°. Tabulate your conclusions from the above exercises.

Practical Exercises introducing Theorems 21 and 22.

(1) Let a straight line cut two other parallel straight lines ; measure any pair of alternate angles. What do you find?

(2) Let a straight line cut two other parallel straight lines ; measure any pair of corresponding angles. What do you find?

(3) Let a straight line cut two other parallel straight lines ; measure the two interior angles on the same side of the line and find their sum. What do you learn?

(4) Let a straight line cut two other parallel straight lines ; measure (*a*) the alternate angles, (*b*) the corresponding angles, (*c*) the two interior angles on the same side of the cutting line. Tabulate your results and state your conclusions.

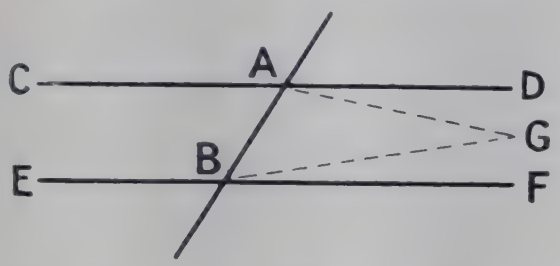
(5) Draw two straight lines AB, CD, each parallel to the straight line EF. By use of setsquares prove whether AB and CD are parallel, or not, to one another. State result.

(6) Repeat Exercise 5, and write down your conclusions from the last two exercises.

Parallel Straight Lines.

Theorem 19.

If a straight line cuts two other straight lines so as to make the alternate angles equal, the two straight lines are parallel.



Let the strt. line **AB** cut two strt. lines **CD**, **EF** at **A** and **B** respectively, so as to make the alt. ∠s **CAB**, **ABF** equal.

Then **CD** and **EF** are parallel.

Proof.—For if **CD** and **EF** are not parallel they must meet when produced either towards **C** and **E** or towards **D** and **F**. Suppose, if possible, they meet when produced towards **D** and **F**, and let **G** be their point of intersection.

Then **GAB** is a △ which has side **GA** produced to **C**,

∴ ext. ∠ **CAB** > int. opp. ∠ **ABF**.

But this is contrary to hypothesis;

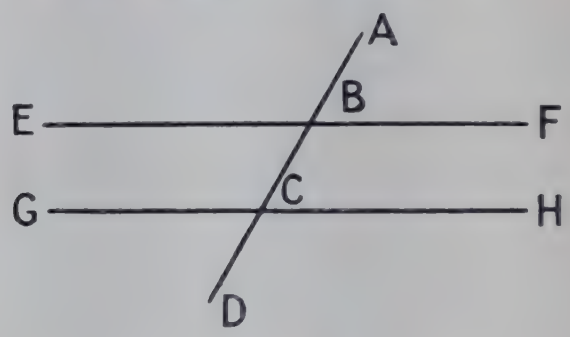
∴ **CD**, **EF** do not meet when produced towards **D** and **F**.

Sim$^{ly.}$ they do not meet when produced towards **C** and **E**;

∴ **CD** and **EF** are parallel. Q.E.D.

Theorem 20.

If a straight line cutting two other straight lines makes
(a) *two corresponding angles equal,*
(b) *the interior angles on the same side of the line supplementary, then the two straight lines are parallel.*



Let the strt. line AD cut two strt. lines EF and GH at B and C.

(*a*) If the corresponding ∠s ABF, BCH are equal,

Then EF is || GH.

PROOF.— ∵ EF cuts AC at B,

∴ ∠ ABF = vert. opp. ∠ EBC (I. 3);

But ∠ ABF = ∠ BCH (Hyp.);

∴ alt. ∠ EBC = alt. ∠ BCH;

∴ EF, GH are parallel (I. 19).

(*b*) If the ∠s FBC, BCH on the same side of the EF are supplementary,

Then EF is || GH.

PROOF.— ∵ the adj. ∠s EBC, FBC = 2 rt. ∠s (I. 1)

and the ∠s FBC, BCH = 2 rt. ∠s (Hyp.),

∴ ∠s EBC, FBC = ∠s FBC, BCH;

∴ ∠ EBC = ∠ BCH;

and these are alt. ∠s;

∴ EF and GH are parallel. Q.E.D.

A straight line which cuts two or more parallel straight lines is called a **transversal.**

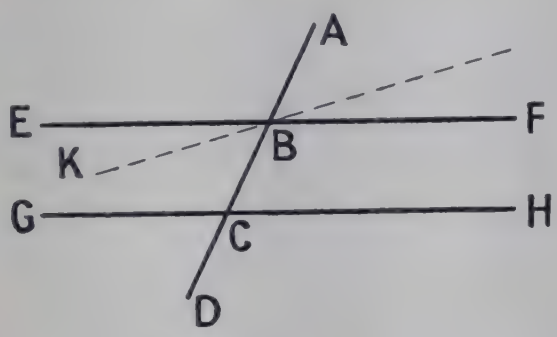
Theorem 21.

If a straight line cuts two other parallel straight lines it makes—

(1) *the alternate angles equal,*

(2) *the corresponding angles equal,*

(3) *the two interior angles on the same side of the line supplementary.*



Let the strt. line ABCD cut two || strt. lines EF, GH at B and C.

Then— (1) alt. ∠ EBC = alt. ∠ BCH.

(2) ∠ ABF = corresp. ∠ BCH.

(3) int. ∠s FBC, BCH = 2 rt. ∠s.

CONSTRUCTION.—If ∠ EBC is not = ∠ BCH, let BK be drawn, making ∠ CBK = ∠ BCH.

PROOF.— Then KB is || GH.

But EF is || GH;

∴ two intersecting strt. lines EF, KB are both || to GH.

But this is impossible (Playfair's Axiom);

∴ ∠ EBC = ∠ BCH.

(2) Again, ∴ ∠ EBC = ∠ BCH (proved),

and ∠ EBC = vert. opp. ∠ ABF;

∴ ∠ ABF = ∠ BCH.

(3) Further, ∵ ∠ EBC = ∠ BCH,

∴ ∠s FBC, EBC = ∠s FBC, BCH.

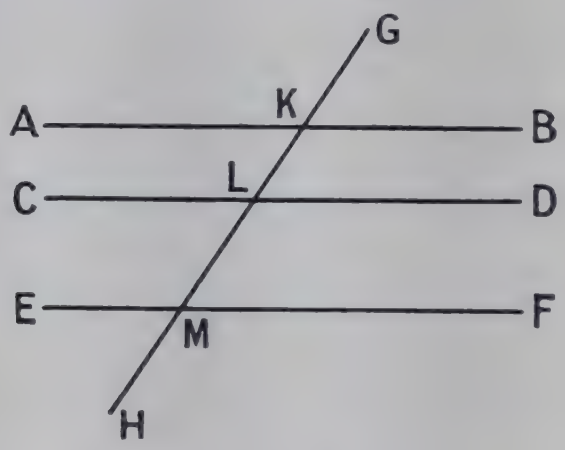
Now ∠s FBC, EBC = 2 rt. ∠s (I. 1).

∴ ∠s FBC, BCH = 2 rt. ∠s. Q.E.D.

COROLLARY.—Two angles whose arms are parallel and drawn in the same sense are equal.

Theorem 22.

Straight lines which are parallel to the same straight line are parallel to one another.



Let the strt. lines AB, CD be each || to the strt. line EF.

Then AB is || to CD.

CONSTRUCTION.—Draw a strt. line GH to cut AB, CD, EF at K, L, M respectively.

PROOF.—∵ EH cuts the parallels AB, EF,

∴ ∠ AKM = alt. ∠ KMF (I. 21).

Again, ∵ GH cuts the parallels CD, EF,

∴ ∠ KLD = corresponding ∠ KMF (I. 21);

∴ ∠ AKM = ∠ KLD;

and these are alt. ∠s,

∴ AB is || to CD (I. 19). Q.E.D.

Exercises.

(1) Let ABC be any triangle. Produce BA to D and CA to E, making AD equal to AB and AE to AC; join DE. Show that the triangle ADE is in all respects equal to the triangle ABC, and that if BC and ED were produced ever so far in either direction they would never meet.

(2) Straight lines which are perpendicular to the same straight line are parallel to one another.

(3) If a line which bisects the exterior vertical angle of a triangle be parallel to the base, prove that the triangle is isosceles.

(4) The side BC of a triangle ABC is bisected in D. DF is drawn parallel to CA to meet AB in F, and DE is drawn parallel to AB to meet CA in E. Join EF. Show that the four triangles AEF, BFD, CDE, and DEF are equal in all respects.

(5) Through a given point draw a straight line, making with the given straight line an angle equal to a given angle. How many such lines can be drawn?

(6) Through a given point draw a straight line, such that the part intercepted between the two parallels may be equal to a given straight line GH.

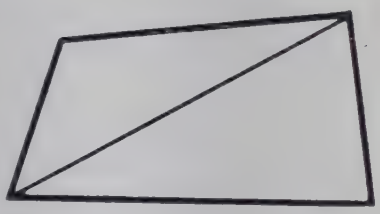
(7) Theorem 22 is the converse of Playfair's axiom. Demonstrate the truth of this theorem, using Playfair's axiom.

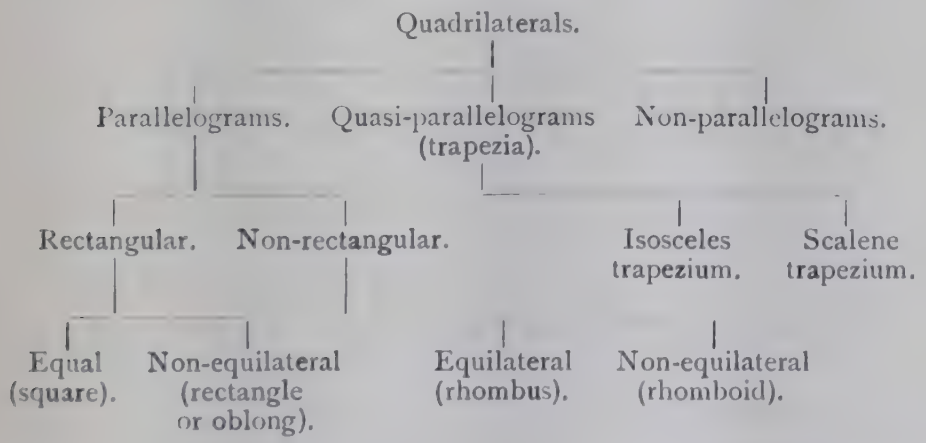
Definitions.

Parallelograms.

A **quadrilateral** is a plane figure bounded by four straight lines.

A **diagonal** is the straight line which joins opposite angular points in a quadrilateral.





A **parallelogram** is a four-sided figure consisting of two pairs of parallel sides.

A **rectangle**, or **oblong**, is a parallelogram which has one of its angles a right angle.

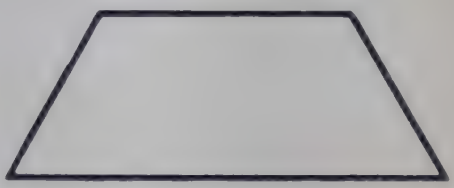
(Euclid's definition is redundant, as it may easily be proved that if one angle in the rectangle is a right angle, then all its angles are right angles.)

The Greek word ἑτερόμηκες, oblong, is expressive of the inequality of adjacent sides.

A **square** is a four-sided figure which has all its sides equal, and one of its angles a right angle.

A **rhombus** is a four-sided figure which has all its sides equal, but its angles are not right angles.

A **rhomboid** is a four-sided figure consisting of two pairs of parallel sides, but its angles are not right angles.



A **trapezium** is a four-sided figure which has two of its sides parallel.

A trapezium which has the sides that are not parallel equal to one another, is called an **isosceles** trapezium.

A trapezium which has its four sides unequal is a **scalene** trapezium.

Practical Exercises introducing Theorem 23.

(1) Draw a parallelogram, using setsquares.

(2) On a piece of paper construct a parallelogram; draw its diagonal; cut out the parallelogram along the diagonal dividing the parallelogram into two triangles. By superposition of one triangle on the other tell what you learn.

(3) Construct a parallelogram ABCD and let its diagonals

intersect at O ; measure OA, OC, OB, OD. Give your observations. Tabulate the conclusions you have obtained from the last three exercises.

(4) Draw two equal and parallel straight lines ; join their corresponding extremities. What figure have you constructed? Give reasons for answer.

(5) Construct a parallelogram having one of its angles a right angle ; measure the other angles and compare their values. What do you conclude?

(6) Prove that a square is equiangular.

Parallels.

Practical Exercises introducing Theorem 24.

(1) Draw three parallel straight lines so that the intercepts made by them on a given straight line that cuts them are equal ; prove by measuring that the corresponding intercepts on any other straight line that cuts them are equal.

(2) Show that a line drawn through the middle point of a non-parallel side of a quasi-parallelogram parallel to the parallel sides bisects the opposite non-parallel sides.

(3) Divide a straight line AB 2·4 in. long into five equal parts, using method as in Problem 14.

(4) Divide a straight line AB 3·2 in. long into seven equal parts (see Problem 15).

(5) Prove experimentally that the straight line drawn through the middle point of a side of a triangle parallel to the base, bisects the remaining side.

(6) Prove that the straight line which joins the middle points of two sides of a triangle is equal to half the third side.

Exercises.

(1) Draw two parallel straight lines at a distance of 3·5 cm. apart.

(2) Draw two parallel straight lines 4·6 in. and 5·8 in. long ; join their extremities towards the same parts. What kind of a quadrilateral is formed?

(3) Draw a triangle with sides 2·4 in., 1·8 in., and 1·6 in., and through each angular point draw a straight line parallel to the opposite side.

(4) Construct a parallelogram with adjacent sides 5·6 cm. and 4·3 cm. and one angle 65°.

(5) Make a rhombus with sides 4·9 cm. and one angle 50°.

(6) Make an angle of 45°. Find a point which is 3 cm. from each arm of the angle.

(7) Construct an equilateral triangle having an altitude of 5 cm.

(8) AB is a given straight line and C a given point on it. Find a point D which shall be 3·4 cm. from AB and 4·8 cm. from C.

(9) Construct a triangle having the base 7·8 cm., one side 6 cm., and a height of 4·4 cm.

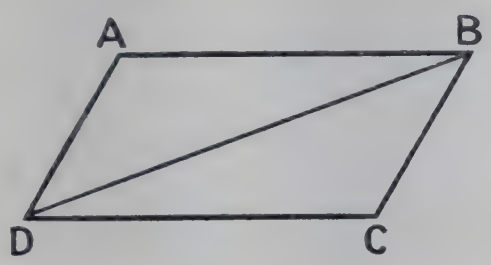
(10) Construct a rhombus with sides 6·4 cm. and one angle 135°, draw the diagonals, and prove that they bisect one another at right angles.

(11) Draw an angle BAC of 64°. Bisect it by AX, and on the bisector cut off AD 3·1 in. Through D draw DY parallel to BA, cutting AC at Y. Prove that the triangle AYD is isosceles.

(12) With the setsquare draw two parallel straight lines about 1½ in. apart. On them take any points A and B. Bisect AB at O. Through O draw three straight lines terminated by the parallels. Prove that all the lines are bisected at O.

Theorem 23.

The opposite sides and angles of a parallelogram are equal, and each diagonal bisects the parallelogram, and the diagonals bisect one another.



Let ABCD be a parallelogram, and BD one of its diagonals.

Then— (1) AB = CD, and AD = CB.
(2) ∠ BAD = ∠ BCD.
(3) ∠ ADC = ∠ CBA.
(4) △ ABD = △ BCD.

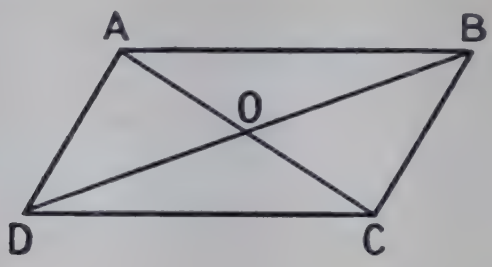
PROOF.—∵ AB is || to DC and BD meets them,
∴ ∠ ABD = alt. ∠ CDB (I. 21);
∵ AD is || to BC and BD meets them,
∴ ∠ ADB = alt. ∠ CBD (I. 21);
∴ in △s ABD, CDB
∠ ABD = ∠ CDB,
∠ ADB = ∠ CBD,
and BD is common to both;
∴ △s ABD, CDB are congruent (I. 13).

∴ AB = CD
and AD = CB } . . . . . (1)

and ∠ BAD = ∠ BCD
Sim^ly. ∠ ABC = ∠ CDA } . . . (2)

And ∵ △ ABD = △ CDB,
∴ BD bisects the ▱^m
Sim^ly. AC bisects the ▱^m . . . (3)

The diagonals of a parallelogram bisect one another.



Let the diagonals AC, BD of $\square^{m}$ ABCD intersect at O,

Then AO = OC, and DO = OB.

PROOF.—$\because$ AB is $\parallel$ to CD and BD meets them,

$\therefore \angle$ OBA = $\angle$ ODC (I. 21);

$\therefore$ in $\triangle$s AOB, DOC

$\angle$ OBA = $\angle$ ODC,

$\angle$ AOB = vert. opp. $\angle$ DOC (I. 3).

and AB = DC (I. 23);

$\therefore \triangle$s AOB, DOC are congruent (I. 13);

$\therefore$ AO = OC,

and DO = OB. Q.E.D.

COROLLARIES.—(1) The distance between parallel straight lines is everywhere the same.

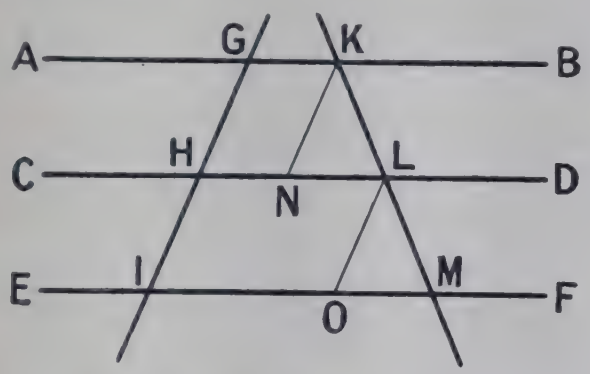
(2) If a parallelogram has one of its angles a right angle, all its angles are right angles; *i.e.* all the angles of a rectangle are right angles.

(3) A square is equilateral and equiangular.

Parallel Straight Lines.

Theorem 24.

If there are three or more parallel straight lines, and the intercepts made by them on any straight line that cuts them are equal, then the corresponding intercepts on any other straight line that cuts them are equal.



Let the parallels AB, CD, EF be cut by the strt. lines GHI, KLM, and let the intercepts GH and HI be equal.

Then the corresponding intercepts KL, LM are equal.

Construction.—Through K and L draw KN, and LO ‖ to GI.

Proof.—∵ CD and EF are ‖ and KM meets them,

∴ ∠ KLN = ∠ LMO (I. 21);

∵ KN and LO are each ‖ to GI,

∴ KN is ‖ to LO (I. 22);

∴ ∠ NKL = corresp$^{g.}$ ∠ OLM (I. 21).

Now GN, HO are $\square^{ms}$,

∴ KN = GH and LO = HI;

But GH = HI (Hyp.);

∴ KN = LO;

∴ in △s KNL, LOM,

∠ KLN = ∠ LMO,

∠ NKL = ∠ OLM,

KN = LO;

∴ △s KNL, LOM are congruent (I. 13);

∴ KL = LM. Q.E.D.

Exercises.

(1) Straight lines bisecting two adjacent angles of a parallelogram intersect at right angles.

(2) No two straight lines drawn from the extremities of the base of a triangle to the opposite sides can possibly bisect each other.

(3) If one angle of a parallelogram is a right angle, all its angles are right angles.

(4) If the opposite sides of a quadrilateral are equal, it is a parallelogram.

(5) If the straight line joining two opposite angles of a parallelogram bisect the angles, the four sides of the parallelogram are equal.

(6) Straight lines bisecting two adjacent angles of a parallelogram intersect at right angles.

(7) ABCD is a parallelogram, and E, F, the middle points of AD and BC respectively. Show that BE and DF will trisect the diagonal AC.

(8) If the diagonals of a quadrilateral bisect one another, the figure is a parallelogram.

(9) If the diagonals of a parallelogram be equal, all its angles are equal.

(10) If the opposite angles of a quadrilateral are equal, it is a parallelogram.

(11) The diagonals of a parallelogram bisect each other.

(12) Any straight line drawn from the vertex of a triangle to the base is bisected by the straight line which joins the middle points of the other sides of the triangle.

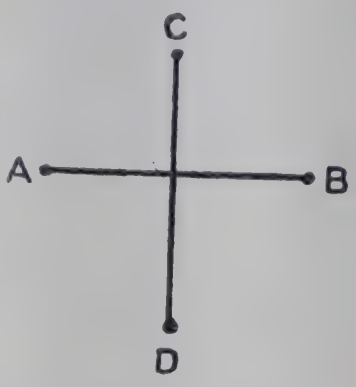
Loci.

If a line or groups of lines satisfy a certain geometrical condition, and if all the points satisfying the given condition lie on that line or group of lines, then the line or group of lines is called the **locus of a point** satisfying that given condition.

Locus of points is sometimes used for *locus of a point*, though the latter form is much to be preferred. Sometimes, in determining a locus, it is thought sufficient to restrict the definition to its first part, sometimes to the second part, although a strict demonstration would need the application of both parts.

EXAMPLE.—*To find the locus of a point equidistant from two given points* A, B.

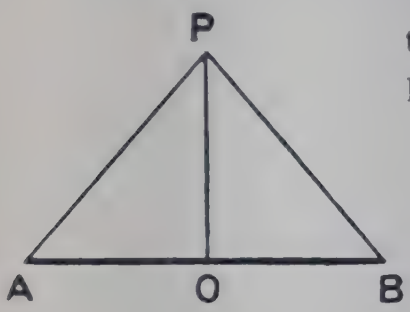
(*a*) Let CD bisect AB at right angles.



Then it can be proved that any point on CD is equidistant from A and B.

∴ we conclude CD is the locus sought.

(*b*) Let P be any point of the locus, and O the middle point of the line AB.



Join OP.

Then, since AP and PB are equal by hypothesis, and OA and OB are equal by construction,

Therefore the angles AOP, BOP are right angles.

Hence the locus of AB is a straight line bisecting AB at right angles.

A point may be subject to two geometrical conditions which completely determine its position.

EXAMPLE 1. — A point is, firstly, to be on AB; and secondly, equidistant from A and B. There is only one point which satisfies these geometrical conditions. But suppose only one geometrical condition to which the point is subject is that it is to be equally distant from A and B, then any

number can be formed, and found satisfying that condition. These will lie on a certain line. This line is called the locus of a point, equidistant from A and B.

EXAMPLE 2.—We may define a circle as a locus of points which are at a given distance from a given point.

Practical Exercises on Loci.

(1) Find the locus of a point which moves so that it is 1·5 in. from a given straight line. What is the locus?

(2) Find the locus to a given point 3·4 cm. from it. What is the locus?

(3) Find the locus of a point equidistant from two given points. What is the locus? Test your result.

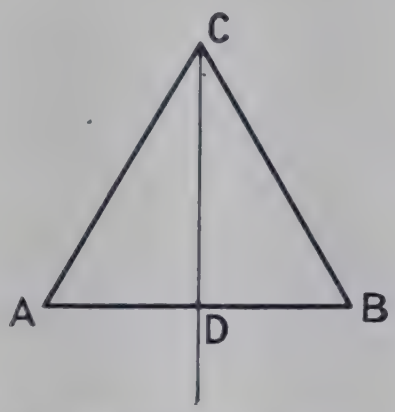
(4) Find the locus of a point equidistant from two intersecting straight lines.

(5) Find the locus of the mid-points of a system of parallel straight lines (chords) to a circle. Prove your result by measurements.

(6) Describe a circle passing through the vertex of a triangle. Find the locus of a point equidistant from A, C; B, C; A, B.

Theorem 25.

The locus of a point which is equidistant from the fixed points is the perpendicular bisector of the straight line joining the two fixed points.



Let A and B be the given points, and C any one position of a point such that CA = CB.

Then C lies on the ⊥ bisector of AB.

CONSTRUCTION.—Bisect AB at D. Join DC.

PROOF.—In △s ADC, BDC

CA = CB (Hyp.),

AD = DB (const.),

CD is common ;

∴ △s ADC, BDC are congruent (I. 14) ;

∴ ∠ ADC = its adj. ∠ BDC ;

∴ CD is ⊥ to AB ;

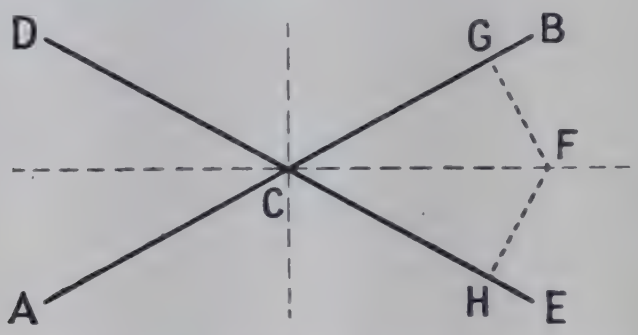
∴ C lies on the ⊥ bisector of AB.

Sim^{ly} it may be proved that every point equidistant from A and B lies on the ⊥ bisector of the strt. line joining AB.

Q.E.D.

Theorem 26.

The locus of a point which is equidistant from two intersecting straight lines consists of the pair of straight lines which bisect the angles between the two given lines.



Let AB, DE be the given strt. lines intersecting at C, and F any such pt. that the ⊥ FG = ⊥ FH.

Then F lies on one of the bisectors of the angles formed by DCE, ACB.

CONSTRUCTION.—Join F to C, the intersection of AB, DE.

PROOF.—In △s FGC, FHC

∠ FGC = ∠ FHC (rt. ∠ s),

FG = FH (Hyp.),

FC is common,

∴ △s are congruent (I. 17);

∴ ∠ GCF = ∠ HCF,

i.e. ∠ BCF = ∠ ECF.

∴ F lies on the bisector of the ∠ BCE. If F be within the ∠ DCB, it may be shown that the point lies on the bisector of the ∠ BCD. Q.E.D.

Exercises.

(1) Find a point which is equidistant from two given intersecting straight lines, and at a given distance from a fixed point.

(2) Find a point which is equidistant from the three sides of a triangle.

(3) Find a point on the base of a scalene triangle equidistant from the two sides.

(4) A and B are two points 3 in. apart; find a point or points distant 4 in. from A and 5 in. from B.

(5) On a given base as hypotenuse right-angled triangles are described. Find the locus of their vertices.

(6) What is the locus of the tip of a clock finger.

(7) A cyclist rides round a circular track, always keeping two yards from the inner edge. What is his locus?

(8) A boy walks along a straight road, keeping to the middle of the road. What is his locus?

(9) Find a point equidistant from the three sides of a triangle.

(10) Find a point on the base of a scalene triangle equidistant from the two sides.

(11) A point E moves along a straight line FD. Find the position in which it is equidistant from two given points A and B.

(12) On a given base construct a triangle of given altitude (2·4 in.) having its vertex on a given straight line.

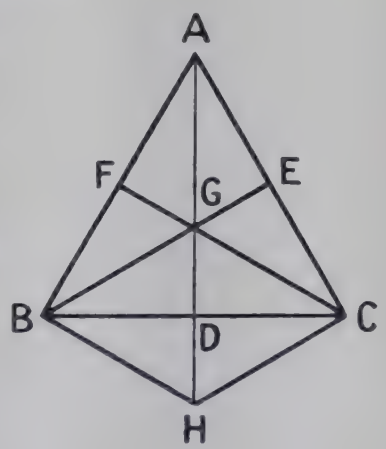
Concurrent Lines.

The **median** of a triangle is a straight line which joins an angular point to the middle point of the opposite side.

The medians of a triangle are concurrent.

Let medians BE, CF of △ ABC be two of its medians, and let them intersect at G. Join AG.

Then AGD is the remaining median of the △.



Construction.—Though C draw CH || to BE; produce AD to meet CH at H. Join BH.

Proof.—In △ AHC,

∵ E is the middle pt. of AC, and EG is || to CH,

∴ G is the middle pt. of AH.

Again, in △ ABH,

∵ F and G are the middle pts. of AB, AH,

∴ FG is || to BH,

i.e. GC is || to BH,

and the fig. BHCG is a $\square^{m}$;

∴ its diagonals bisect each other (I. 23);

∴ BD = DC;

∴ AGD is the third median,

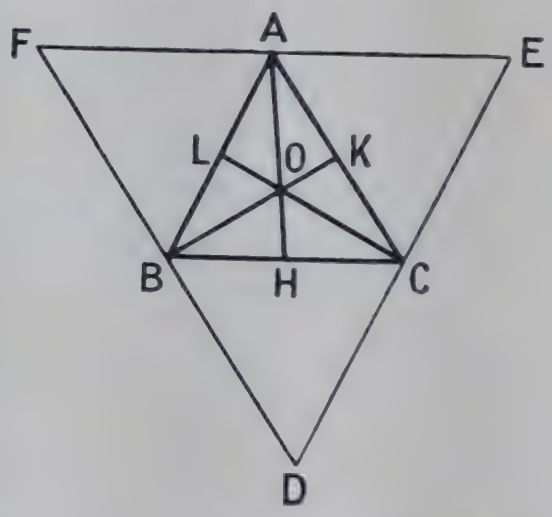
and the three medians meet at the pt. G. Q.E.D.

The point of concurrency of the three medians is called the **centroid** of the triangle.

COROLLARY.—Since D is the middle pt. of GH (proved),

$$GD = \tfrac{1}{2} GH$$
$$= \tfrac{1}{2} GA ;$$
$$\therefore GD = \tfrac{1}{3} AD$$
$$\text{or } AG = \tfrac{2}{3} AD.$$

The perpendiculars from the vertices of a triangle to the opposite sides are concurrent. (*The altitudes of a triangle are concurrent.*)



In △ ABC suppose AH, BK, CL ⊥ BC, CA, AB respectively.

Then AH, BK, CL are concurrent.

Through the corners of △ ABC draw strt. lines || to opp. sides, forming a new △ FED.

PROOF.— AD is a ▱^{m} (Const.).

BD = AC (I. 23).

CF is a ▱^{m} (Const.).

BF = AC (I. 23),

∴ BD = BF,

∴ B is the middle pt. of FD,

Sim$^{ly.}$ A „ „ „ FE,

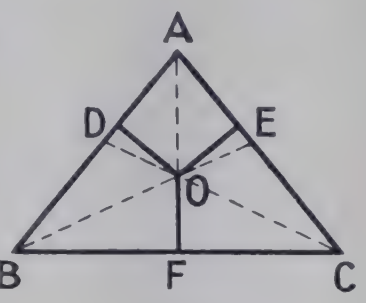
C „ „ „ DE,

and AH, BK, CL are the perpendiculars bisecting FE, FD, DE ;

∴ AH, BK, CL are concurrent at the point O, which is called the **orthocentre** of the △ ABC. Q.E.D.

The straight lines which bisect the angles of a triangle are concurrent.

In △ ABC let the strt. lines BE, CD bisect the ∠s ABC, ACB respectively.



Join AO, the pt. of intersection of BE, CD,

Then AO bisects ∠ BAC.

CONSTRUCTION.—From O draw OD, OE, OF ⊥ BA, CA, BC respectively.

PROOF.—∵ BO bisects ∠ ABC,

∴ it is the locus of a pt. equidistant from BA and BC;

∴ OD = OF (I. 26).

Sim$^{ly.}$ OE = OF,

∴ OE = OD;

∴ O is on the locus of a pt. equidistant from AB and AC;

∴ OA bisects ∠ BCD,

and the bisectors are concurrent at pt. O. Q.E.D.

DEFINITIONS.

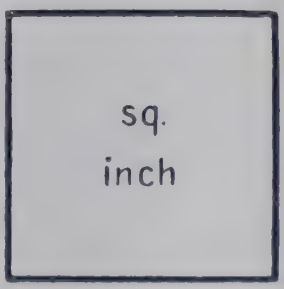
Areas.

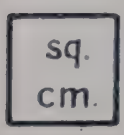
Area of a figure is the amount of surface enclosed by its boundary.

To measure areas numerically some unit of area must be fixed upon. In experimental geometry a square inch or square centimetre is taken as the unit of area.

A **unit of area** is a square with a side of unit length.

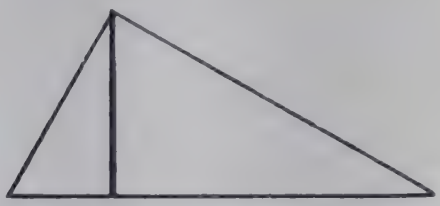
A **square inch** (sq. in.) is a square with a side one inch in length.





A **square centimetre** (sq. cm.) is a square with a side one centimetre in length.

The **altitude**, or height, of a triangle is the perpendicular drawn from the vertex to the base.



The **altitude**, or height, of a parallelogram is the perpendicular on any side called the base from any point in the opposite side.

A rectangle is said to be **contained by** a pair of adjacent sides.

Area of a Rectangle.

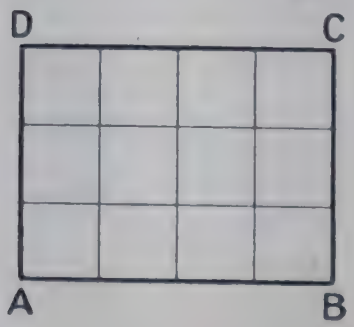
To prove the rule for finding the area of a rectangle.

Let **ABCD** be a rectangle whose length **AB** and whose breadth **AD** is four and three units of length respectively.

Divide **AB** into four equal parts and **AD** into three equal parts.

Through the points of section in each line draw || lines, as shown in figure.

Now, area of rectangle = 12 units of area
= 4 × 3 units of area
= **AB** × **AD**
= **length** × **breadth.**



Area of Parallelograms.

Practical Exercises introducing Theorem 27.

(1) Construct a parallelogram ABCD, having AB 7·8 cm. long, AD 5·6 cm. long, and the angle at A 65°.

(2) Describe two parallelograms on the same base and of the same altitude as in Fig. 1. Compare the areas of the three triangles and of the two parallelograms; state clearly the reasons for answer.

Fig. 1.

(3) (*a*) Draw any two parallelograms ABCD, ABEF on a piece of plain paper; cut out the triangle CFB, and apply it on the triangle DEA. What do you observe about the areas of the triangles and of the two parallelograms?

(*b*) Make a tracing of the triangle CFB, and by superposition compare the areas of the triangles CFB, DEA, etc., and of the two parallelograms.

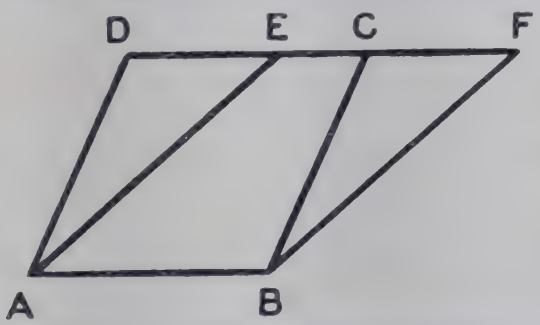


Fig. 2.

(4) Repeat Exercise 3, drawing the parallelograms as shown in Fig. 3. Compare the areas of the two triangles EBC, DFA, and of the two parallelograms.

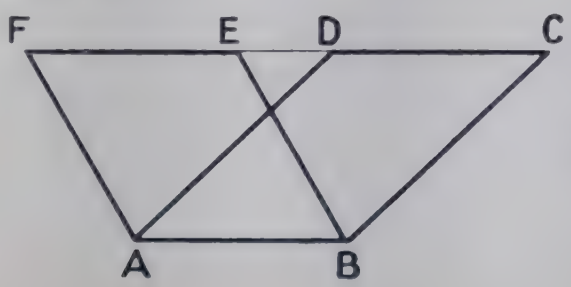


Fig. 3.

(5) To find the area of a parallelogram. On the base AB of the parallelogram ABEF make the rectangle ABCD. Making use of the conclusion drawn from Exercise 4, determine the rule for finding the area of a parallelogram.

(6) Find the area of a parallelogram ABCD, given the base AB 10·5 cm. and the altitude 6·4 cm.

Area of Triangles.

Practical Exercises introducing Theorem 28.

(1) Construct a triangle, ABC, and a rectangle on the same base and on the same side of it, having equal altitudes.

(2) Draw two triangles ABC, ABD on the same base AB and of equal altitudes. Compare their areas, and give reasons for answer.

(3) Make two triangles ABC, DEF on bases AB and EF and of equal altitudes. Prove that the triangles are equal in area. State clearly what you learn from the foregoing exercises.

(4) Construct two triangles on equal bases along the same straight line, and having a common vertex; prove that they are equal in area.

(5) Draw any triangle ABC; construct the parallelogram of which it is half. Deduce the rule for finding the area of a triangle.

(6) Find the area of a triangle, given the three sides 6·4 cm., 7·5 cm., 8·3 cm.

Practical Exercises introducing Theorem 29.

(1) Draw any rectangle ABCD, and draw its diagonals to form the triangles ABC, ABD. Are the triangles equal in area? Are their altitudes equal?

(2) Draw any parallelogram ABCD, and draw its diagonals to form the equal triangles ABC, ABD. Why are they equal? Compare the heights of the triangles.

(3) Construct two triangles ABC, ABD on the same base and equal in area, using any method you like.

(4) Make any two triangles ABC, ABD on the same base and equal in area, and compare their altitudes. What do you conclude from the above exercises?

(5) Construct two equal triangles on equal bases.

(6) Draw two equal triangles on the same or equal bases, and prove that they are between the same parallels.

Practical Exercises.

(1) Make a rhombus equal to a given parallelogram.

(2) Construct an isosceles triangle equal to a given triangle.

(3) Bisect a triangle by a line drawn through a point in one of its sides.

(4) Prove by measurement that the perimeter of an isosceles triangle is less than that of an equal triangle on the same base.

(5) Show that if two equal triangles be on the same base, but on opposite sides of it, the straight line which joins their vertices will be bisected by the base.

(6) The straight line which joins the two middle points of two sides of a triangle is parallel to the third side and one half of it. Prove this statement.

(7) Construct a rectangle equal to a given triangle.

(8) Divide a triangle into four equivalent triangles.

(9) Draw a parallelogram of area 30 sq. cm., base 6 cm., and angle 75°. Give the length of the altitude and the other sides.

(10) Transform a parallelogram with an angle of 85° into an equivalent parallelogram with angle of 65°.

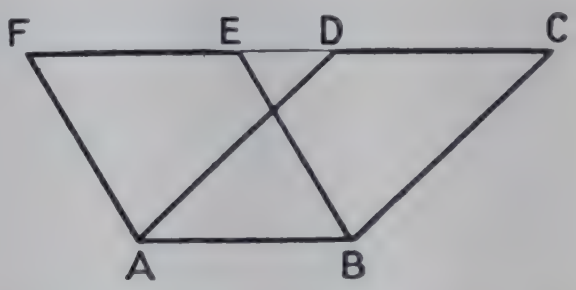
(11) Divide a parallelogram into three equal parallelograms.

(12) Transform a given triangle into an equivalent right-angled triangle.

Area of Parallelograms.

Theorem 27.

Parallelograms on the same base and of the same altitude are equal in area.



Let **ABCD, ABEF** be $\square^{ms}$ on the same base **AB** and of the same altitude.

Then the $\square^{m}$ **ABCD** = $\square^{m}$ **ABEF** in area.

Proof.—$\because$ the altitudes of the $\square^{ms}$ are equal (Hyp.),

$\therefore$ they are between the same parallels (I. 23, Cor.);

$\therefore$ **CDEF** is a straight line.

In the $\triangle$s **ADF, BCE**

AF = **BE** (I. 23),

$\angle$ **ADF** = $\angle$ **BCE** (I. 21),

$\angle$ **AFD** = $\angle$ **BEC** (I. 21),

$\therefore$ the $\triangle$s **ADF, BCE** are congruent (I. 13).

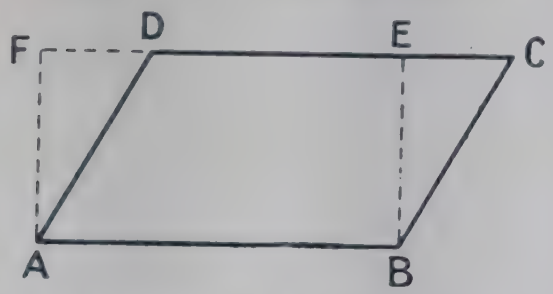
Now if from the whole figure **ABCF** each of these equal $\triangle$s be taken away in turn, the remainders are $\square^{ms}$ **ABCD, ABEF,**

$\therefore$ $\square^{m}$ **ABCD** = $\square^{m}$ **ABEF** in area. Q.E.D.

Corollaries.—(1) Parallelograms on the same base and between the same parallels are equal in area.

(2) Parallelograms on equal bases and of the same altitude are equal in area.

AREA OF A PARALLELOGRAM.



Let ABCD be a $\square^{m}$, and ABEF a rectangle on the same base AB, and of the same altitude AF.

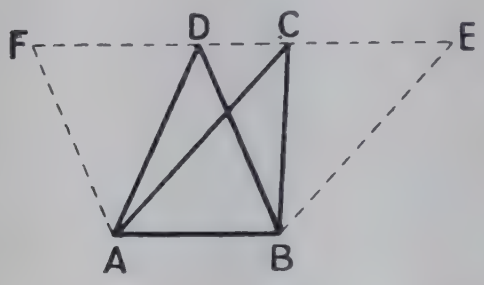
Area of $\square^{m}$ ABCD = area of rect. ABEF
= AB × BE
= base × altitude,

∴ area of a parallelogram = base × altitude.

Area of Triangles.

Theorem 28.

Triangles on the same base and of the same altitude are equal in area.



Let the △s ABC, ABD be on the same base AB, and of the same altitude.

Then △ ABC = △ ABD in area.

Construction. — Join DC; through B draw BE || to AC to meet DC produced in E; through A draw AF || to BD to meet CD produced in F.

Proof.— ▱^{m} ABEC = ▱^{m} ABDF in area (I. 27).

But △ ABC = $\frac{1}{2}$ ▱^{m} ABEC (I. 23),

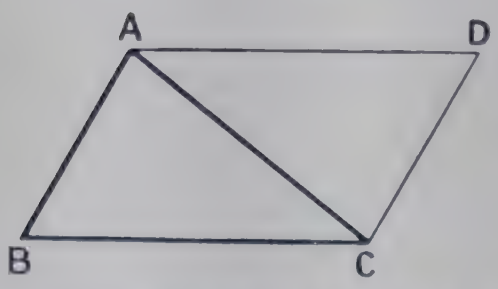
and △ ABD = $\frac{1}{2}$ ▱^{m} ABDF (I. 23);

∴ △ ABC = △ ABD in area. Q.E.D.

Corollaries.—(1) Triangles on equal bases and of the same altitude are equal in area.

(2) Triangles on the same or equal bases and between the same parallels are equal in area.

Area of a Triangle.



Let ABC be a △; complete the ▱^{m} ABCD by drawing AD || to BC, and DC || to AB.

Area of △ ABC

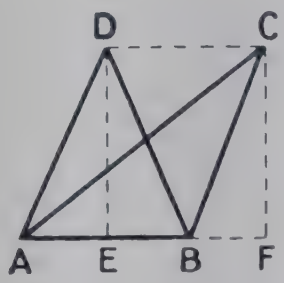
= $\frac{1}{2}$ ▱^{m} ABCD (I. 23)

= $\frac{1}{2}$ base × altitude.

(3). The area of a triangle is half the area of the parallelogram on the same base and having the same altitude.

Theorem 29.

Equal triangles on the same base are of the same altitude.



Let △s ABC, ABD stand on the same base AB and be equal in area; and let ED and FC be their altitudes.

Then ED = FC.

PROOF.—△ ABD = $\frac{1}{2}$ rect. ABED (II. Cor. 28).

△ ABC = $\frac{1}{2}$ rect. ABFC (II. Cor. 28).

As △ ABC = △ ABD in area (Hyp.),

∴ $\frac{1}{2}$ rect. ABED = $\frac{1}{2}$ rect. ABFC;

But AB is common,

∴ ED = FC. Q.E.D.

COROLLARIES.—(1) Equal triangles on equal bases are of the same altitude.

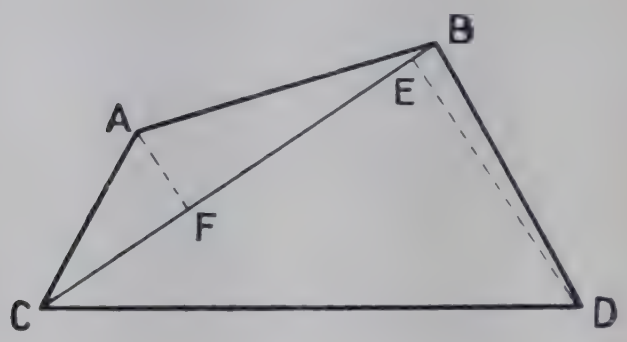
(2) Equal triangles on the same or equal bases are between the same parallels.

AREA OF QUADRILATERALS.

The quadrilateral is divided into two triangles by drawing one of its diagonals. The area of each triangle is got separately by calculation from the given measurements. The sum of the areas of the two triangles is the area of the given figure.

EXAMPLE.—Area of the quadrilateral

$$ABDC = \text{area of } \triangle ABC + \text{area of } \triangle CDB$$
$$= \tfrac{1}{2} CB \,.\, AF + \tfrac{1}{2} CB \,.\, ED.$$



EXAMPLE.—To prove the rule for finding the area of a trapezium.

$$\text{Area of trapezium} = \text{area of } \triangle ABD + \text{area of } \triangle CBD$$
$$= \tfrac{1}{2} AB \,.\, FD + \tfrac{1}{2} CD \,.\, CE$$
$$= \tfrac{1}{2} FD(AB + CD),$$
$$\therefore CE = FD$$
$$= \tfrac{1}{2} \text{altitude} \times \text{sum of the parallel sides}.$$

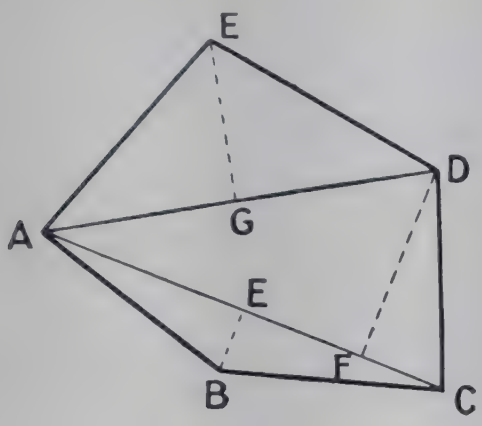
AREA OF ANY RECTILINEAL FIGURE.

The rectilineal figure is divided into triangles by joining any vertex to the remaining vertices. The area of each triangle is got separately by calculation from the given measurements. The sum of the areas of the triangles is the area of the rectilineal figure.

EXAMPLE.—Area of the rectilineal figure

ABCDE = area of △ ABC + area of △ ACD + area of △ ADE

$= \frac{1}{2}$ AC . EB + $\frac{1}{2}$ AC . FD + $\frac{1}{2}$ AD . GE.



Numerical Exercises on Areas.

(1) Find the areas of parallelograms from the following data :—

(*a*) base = 27·4 cm., altitude = 14·3 cm.
(*b*) base = 4·8 in., altitude = 3·1 in.
(*c*) base = 9·4 in., altitude = 7·3 in.
(*d*) sides 12·4 cm. and 11·3 cm., angle 65°.
(*e*) sides 14·3 in. and 9·3 in., angle 75°.
(*f*) sides 19·3 cm. and 8·4 cm., angle 52°.

(2) The area of a parallelogram is 28·3 sq. cm. ; the base is 5·6 cm. Find the height.

(3) Find the areas of triangles from the following measurements :—

(*a*) base = 14·3 cm., altitude = 9·4 cm.
(*b*) base = 9·8 in., altitude = 5·7 in.
(*c*) base = 141·6 cm., altitude = 64·3 cm.
(*d*) sides = 2·4 in., 3·6 in., 2·8 in.
(*e*) sides = 7·8 cm., 8·6 cm., 5·6 cm.
(*f*) sides = 8·9 cm., 6·4 cm., 5·4 cm.

(4) Find the area of a triangle, given the base 6·4 cm., and the base angles equal to 55° and 40° respectively.

(5) The area of a triangle is 36 sq. cm.; the base is 4·8 cm. Find the altitude.

(6) The area of a triangle is 120·2 sq. cm. ; the altitude is 16 cm. What is the length of the base?

(7) When you quadruple the length of a rectangle without altering its breadth, how many times do you increase the area?

(8) When you quadruple the length and the breadth, how many times do you increase the area?

N.B.—The rules for finding the area of a rectangle, square, parallelogram, triangle, irregular rectilinear figure [by reduction into triangles] may be applied to areas of walks, pavements, streets, tablecloths, room-floors, walls, doors, pictures, etc. Practice in the application of these rules should be obtained.

Exercises.

(1) Make a rhombus equal to a given parallelogram.

(2) Bisect a triangle by a line drawn through a point in one of its sides.

(3) If two triangles have two sides of the one equal to two sides of the other, each to each, and the angles contained by these sides supplementary, the triangles are equal in area.

(4) If a quadrilateral is bisected by each of its diagonals, it is a parallelogram.

(5) Bisect any quadrilateral by a line drawn through one of its angles.

(6) Describe a right-angled triangle which shall be equal to a given square.

(7) If the four sides of a quadrilateral be bisected, the lines joining these points will form a parallelogram, and shall be half of the quadrilateral.

(8) The straight line which joins the two middle points of two sides of a triangle is parallel to the third side, and one half of it.

(9) The straight line which joins the middle points of two sides of a triangle is parallel to the third side.

(10) A parallelogram is divided by its diagonals into four triangles of equal area.

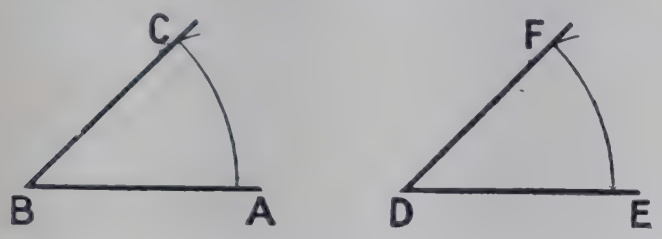
(11) ABCD is a quadrilateral, having BC parallel to AD ; E is the middle point of DC. Show that the triangle AEB is half the quadrilateral.

(12) Bisect a given triangle by a straight line drawn through a given point in a side.

CONSTRUCTIONS.

Problem 1.

To draw an angle equal to a given angle.



Suppose ∠ ABC be the given angle.

Draw a line DE. With C as centre and BA, any length, as radius describe a ⊙ cutting the arms of the ∠ ABC at A and C.

With D as centre, using the same radius BA, draw a ⊙ cutting DE at F.

Take the distance AC in your compasses, and with E as centre describe a ⊙ cutting EF at F.

Join FD.

(*a*) Prove accuracy of the angle by measuring the angles with the protractor.

(*b*) Prove again by joining CA, FE, using Theorem 14.

Exercises.

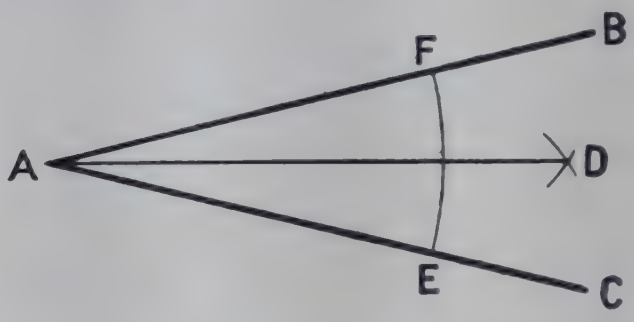
(1) Test the accuracy of your set squares 45°, 60°, 90°, 30°.

(2) By separate construction show the positions of the hands of a clock at the following hours :—

(*a*) 2 o'clock, (*b*) 9 o'clock, (*c*) 4 o'clock, (*d*) 3 o'clock, (*e*) 2.30 o'clock, (*f*) 7.30 o'clock.

Problem 2.

To bisect a given angle.



With A as centre, and any radius, draw an arc of a circle cutting AC and AB in E and F respectively.

With E and F as centres, and any convenient radius, draw arcs cutting in D.

Join AD.

AD bisects ∠ BAC.

Prove theoretically AD bisects ∠ BAC, using Theorem I. 14.

Exercises.

(1) Bisect an angle of 65°.

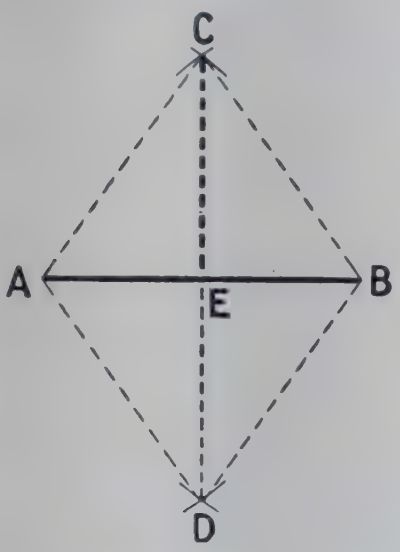
(2) Divide an angle of 120° into four equal parts.

(3) Draw a straight line AOB. At O make an angle AOC = 84°. Bisect each of the angles AOC, BOC, and measure the angle between the bisectors.

(4) Bisect an obtuse angle. Test accuracy by measurement with protractor.

Problem 3.

To bisect a given straight line.



Let AB be the given strt. line.

CONSTRUCTION.—With centres A and B, and convenient radius, describe arcs on each side of AB intersecting at C and D.

Join CD, cutting AB at E,

Then AB is bisected at E.

PROOF.—Join CA, CB, DA, DB.

In △s ACD, BCD,

AC = BC (equal radii),

AD = BD („ „),

and CD is common,

∴ △s ACD, BCD are congruent (I. 14).

Again, in △s ACE, BCE,

AC = BC,

CE is common,

∠ ACE = ∠ BCE (proved),

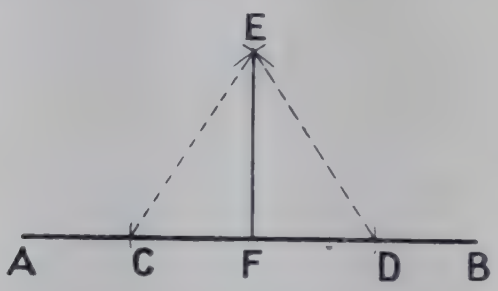
∴ △s ACE, BCE are congruent (I. 7).

∴ AE = BE;

∴ AB is bisected at E. Q.E.F.

Problem 4.

To draw a straight line perpendicular to a given straight line at a given point in it.



Let AB be the given straight line, and F the point in it.

CONSTRUCTION.—From F cut off FC, FD, any two equal lengths, along the given straight line.

With centres C and D, and any convenient radius, describe two arcs cutting at E.

Join EF,

Then EF is $\perp$ AB.

PROOF.— Join EC, ED.

In $\triangle$s CEF, DEF,

CE = DE,

CF = FD,

FE is common,

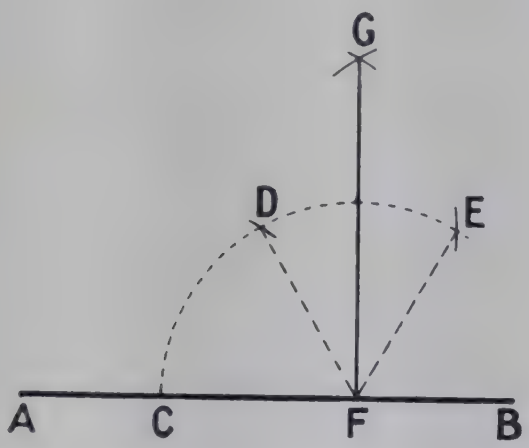
$\therefore \triangle$ CEF = $\triangle$ DEF (I. 14);

$\therefore$ adj. $\angle$ CFE = adj. $\angle$ EFD;

$\therefore$ EF is $\perp$ to AB. Q.E.F.

Problem 5.

To draw a straight line perpendicular to a given straight line at a given point in it.



CONSTRUCTION.—With centre F, and any convenient radius, draw an arc CDE cutting AB in C.

With C as centre, and with the same radius, draw an arc cutting the arc CDE at D.

With D as centre, and with the same radius, draw an arc cutting the arc CDE at E.

Bisect the $\angle$ DFE (Prob. 2).

Then GF is $\perp$ to AB.

PROOF.—$\angle$s CFD, DFE may be considered as angles of an equilateral $\triangle$.

$\therefore \angle$ CFD $= \angle$ DFE $= 60^\circ$,

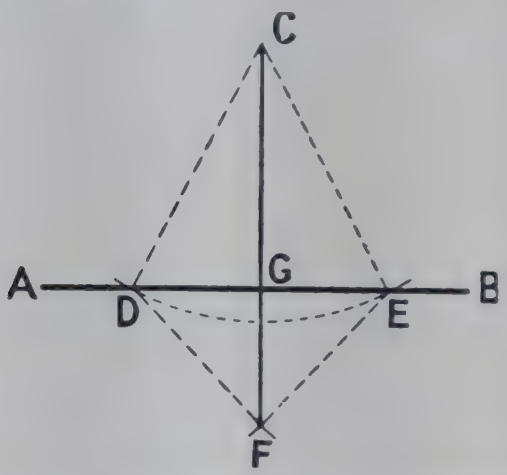
$\angle$ DFG $= \frac{1}{2} \angle$ DFE $= 30^\circ$,

$\therefore \angle$ CFG $= 90^\circ$;

$\therefore$ GF is $\perp$ to AB. Q.E.F.

Problem 6.

To draw a straight line perpendicular to a given straight line from a given point outside it.



Let AB be the given strt. line, and C the given pt. outside it.

CONSTRUCTION.—With centre C, and any convenient radius, describe an arc cutting AB at D and E.

With centres D and E, and any convenient radius, describe equal arcs intersecting at F.

Join CF, cutting AB at G.

Then CG is $\perp$ to AB.

PROOF.—CD = CE, DF = EF.

In $\triangle$s DCF, ECF,

DC = EC (equal radii),

DF = EF,

CF is common,

$\therefore \triangle$ DCF = $\triangle$ ECF (I. 14);

$\therefore \angle$ DCG = $\angle$ ECG.

In $\triangle$s DCG, ECG,

DC = EC,

CG is common,

$\angle$ DCG = $\angle$ ECG (I. 7),

$\therefore \triangle$ DCG = $\triangle$ ECG;

$\therefore$ adj. $\angle$ DGC = adj. $\angle$ EGC;

$\therefore$ CG is $\perp$ to AB. Q.E.F.

Exercises.

(1) Bisect a straight line 6·4 cm. long. Prove that your construction is right.

(2) Bisect any straight line into four equal parts by repeated bisection.

(3) Draw a line half as long again as 4·6 cm.

(4) Draw a straight line 6·8 cm. long, and at a point 3 cm. from one end of it erect a perpendicular to it.

(5) Draw several straight lines freehand, and test accuracy by construction.

(6) A 20-ft. ladder leans against a wall. The bottom of the ladder is 8 ft. from the wall. To what height does it reach?

(7) A pole is 25 ft. long, and casts a shadow of 20 ft. Find the altitude of the sun.

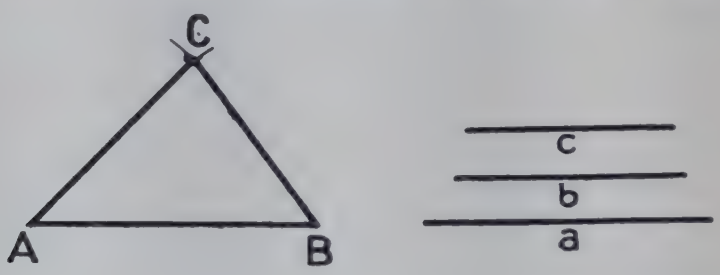
(8) Two men stand on opposite banks of a river 40 ft. wide, directly in front of one another. To the right of one of the men there is a tree 12 ft. away. How far is the other man from the tree?

(9) A ladder stands against a wall. The bottom of it is 16 ft. from the wall, and the top of it touches the wall 24 ft. from the ground. How long is the ladder?

Construction of Triangles.

Problem 7.

To draw a triangle having given the lengths of the three sides.



Let a, b, c be the lengths of the sides of the required $\triangle$.

Draw a strt. line AB $= a$.

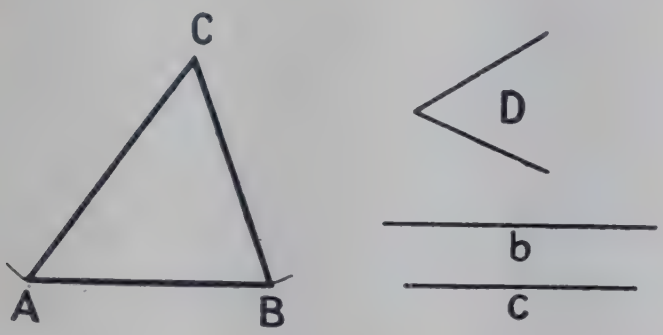
With centres A and B, and radii b and c respectively, draw arcs cutting one another at C.

Join CA, CB.

Then ABC is the required triangle. Q.E.F.

Problem 8.

To describe a triangle having given two sides and the included angle.



Let b, c be the lengths of the two given sides, and D the given angle of the required triangle.

CONSTRUCTION.—At C make $\angle$ ACB = $\angle$ D (Prob. 1).

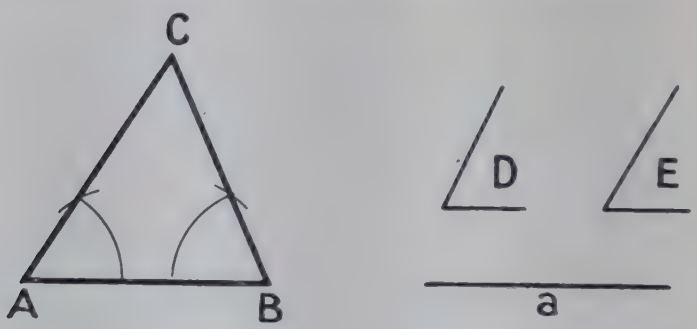
Cut off CA = b, CB = c.

Join AB.

Then ABC is the required $\triangle$. Q.E.F.

Problem 9.

To construct a triangle having given a side and two angles.



Let *a* be the length of the given side, D and E the given angles of the required triangle.

CONSTRUCTION.—Draw a st. line AB = *a*.

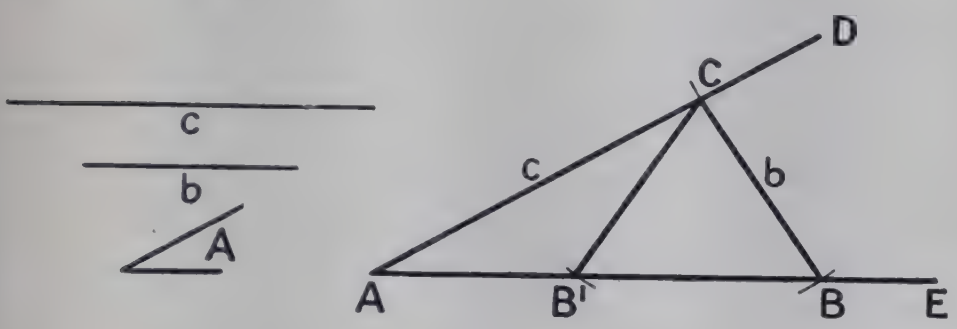
At B make ∠ ABC = ∠ D
At A make ∠ BAC = ∠ E } (Prob. 1).

Produce the sides AC, BC to meet at C.

Then ABC is the required triangle. Q.E.F.

Problem 10.

To construct a triangle having given two sides and an angle opposite to one of them.



Let b, c be the lengths of the given sides, and A the given angle of the required $\triangle$.

CONSTRUCTION.—Draw any strt. line AE; at A make $\angle$ BAC = $\angle$ A.

From AD cut off AC = c.

With centre C, and radius b, draw arcs cutting AE at B and B^1.

Join CB, CB^1.

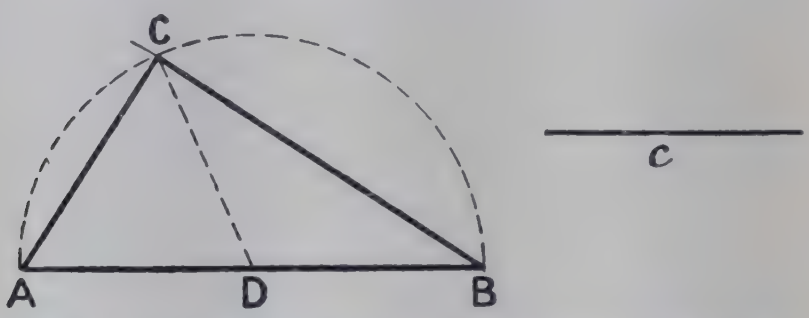
Then either of the $\triangle$s ACB, ACB^1 is the required $\triangle$.

Q.E.F.

This construction, which admits of a double solution, is termed the *Ambiguous case.*

Problem 11.

To describe a right-angled triangle having given the hypotenuse and one side.



Let AB be the hypotenuse, and c the given side of the required $\triangle$.

CONSTRUCTION.—Bisect AB at D. With centre D, and radius DA, describe a semi-$\odot$.

With centre A, and radius c, draw an arc cutting the semi-$\odot$ at C.

Join CA, CB.

Then $\triangle$ ABC is the required $\triangle$.

PROOF.—Join CD.

$\because$ DA = DC (equal radii),

$\therefore \angle$ DCA = $\angle$ DAC (I. 8).

$\because$ DC = DB,

$\angle$ DCB = $\angle$ DBC,

$\therefore \angle$ ACB = $\angle$ CAB + $\angle$ CBA

$= \frac{1}{2}(180^\circ)$. (I. 5)

$= 90^\circ$. Q.E.F.

Miscellaneous Exercises.

(1) Construct a triangle in which—

(a) AB = 6 cm., BC = 7·4 cm., CA = 8·5 cm.
AB = 3·5 in., BC = 2·8 in., CA = 1·7 in.

(b) AB = 6·4 cm., $\angle$ B = 110°, $\angle$ C = 65°.
AB = 3 in., $\angle$ B = 55°, $\angle$ C = 74°.
AB = 5·7 cm., $\angle$ B = 64°, $\angle$ C = 96°.

(*c*) Each of the base angles—
(1) = 65°.
(2) = 44°.
(3) = 72°.

(*d*) AB = 3 in., AC = 4·6 in., included ∠ A = 52°.
AB = 7·5 cm., AC = 5·4 cm., included ∠ A = 40°.
AB = 6·8 cm., AC = 4·8 cm., included ∠ A = 70°.

(2) Draw a right-angled triangle in which one acute angle is 54°. What is the other?

(3) In a triangle ABC make ∠ A = 50°, ∠ B = 74°. What is ∠ C?

(4) Describe an isosceles triangle ABC having a vertical angle of 64°. What are the base angles?

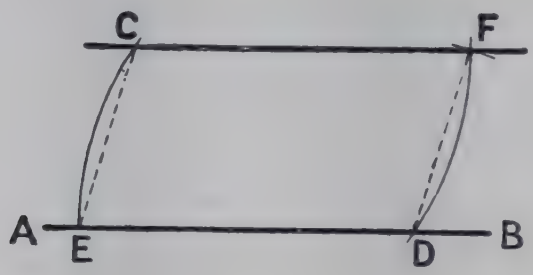
(5) Make a triangle having ∠ A + ∠ B = ∠ C. What is the magnitude of ∠ C?

(6) Describe an equilateral triangle having given the altitude 2·5 in.

CONSTRUCTIONS.

Problem 12.

Through a given point to draw a straight line parallel to a given straight line.



Let AB be the given strt. line, and C the given pt. without it.

CONSTRUCTION.—From D, any pt. in AB, as centre, and radius DC, draw an arc of a ⊙ cutting AB in E. With centre C, and the same radius (DC), draw an arc of a ⊙ DF.

With centre D, and radius = EC, draw an arc of a ⊙ cutting DF in F.

Join CE, FD, CF.

Then CF is || to AB.

PROOF.—In the quadrilateral CEDF,

CE = FD,

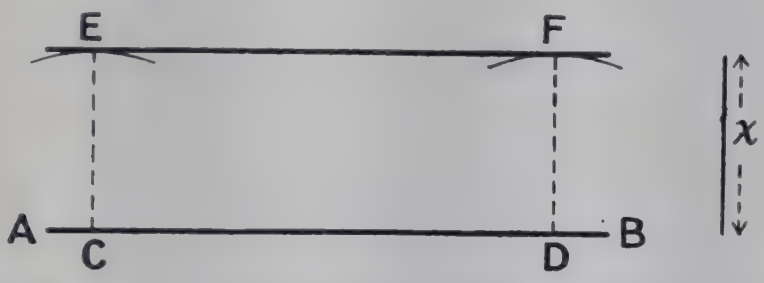
CF = ED,

∴ CEDF is a $\square^{m}$;

∴ CF is || to AB. Q.E.F.

Problem 13.

To draw a straight line parallel to a given straight line and at a given distance from it.



Let AB be the given strt. line, and x the given distance.

CONSTRUCTION.—In AB take any two points C, and D, as near the extremities of the line as convenient. With centre C, and radius x, describe an arc of a ⊙.

With centre D, and radius x, describe an arc of a ⊙.

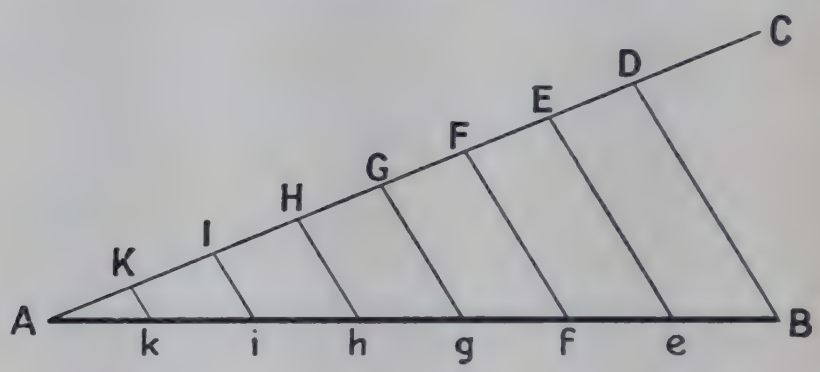
Draw a common tangent EF to the two arcs of ⊙s.

Then EF is || to AB.

Give proof.

Problem 14.

To divide a straight line into any number of equal parts.



Let AB be the given strt. line.

It is required to divide it into, say, seven equal parts.

CONSTRUCTION.—From A draw any strt. line AC; from AC mark off any seven equal lengths, AK, KI, IH, HG, GF, FE, ED.

Join DB, and through E, F, G, etc., draw E*e*, F*f*, G*g*, etc., || to DB.

Then AB is divided into seven equal parts.

PROOF.—AK = KI = IH = HG . . . .

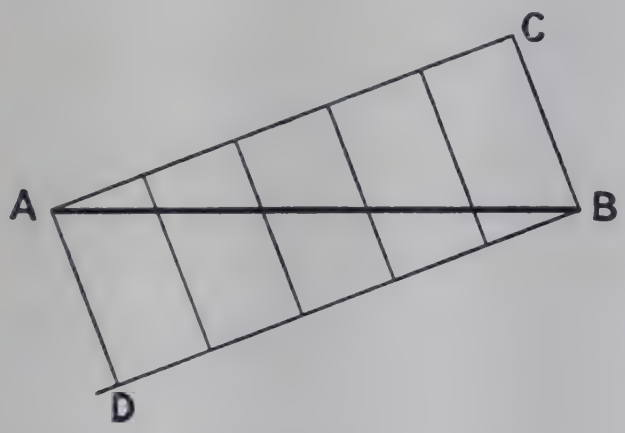
and K*k*, I*i*, H*h*, G*g* . . . . are all ||,

$\therefore$ A*k* = *ki* = *ih* = *hg* = *gf* . . . . (I. 24);

$\therefore$ AB is divided into seven equal parts. Q.E.F.

Problem 15.

To divide a straight line into any number of equal parts (*say five*).



Let AB be the given strt. line.

It is required to divide it into five equal parts.

CONSTRUCTION.—From A draw AC of any length, and mark off any five equal parts.

Through B draw BD || to AC and mark off, commencing at B, five parts, each equal to the parts in AC.

Join the corresponding parts as in diagram.

Then AB is divided into five equal parts.

Give proof.

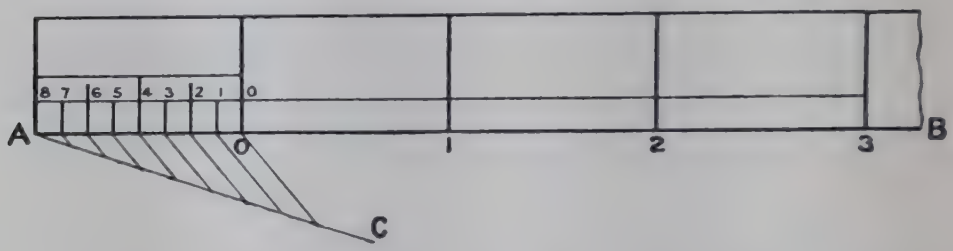
Exercises.

(1) Draw a line 8·7 cm. long, and divide it into six equal parts. Verify your result by measurement and calculation.

(2) Draw a line 4·4 in. long, and cut off two-ninths. Verify your result by measurement and calculation.

Problem 16.

To construct a plane scale (say 1 inch to the mile).



CONSTRUCTION.—Take a line, AB, not less than 4 inches long, and divide it into inches.

From A draw any line AC. Along AC mark off eight equal lengths, commencing at A, and proceed as in Prob. 14.

Exercises.

(1) Draw a line representing five miles three furlongs—scale, one inch to the mile.

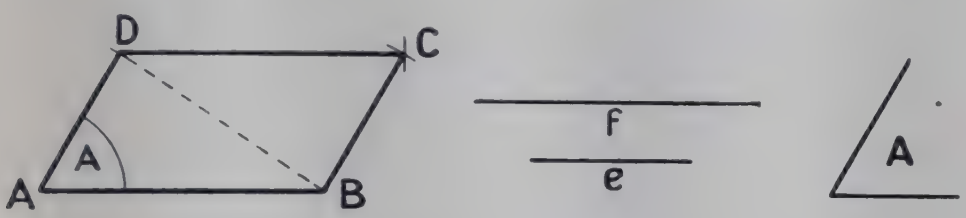
(2) The scale of a map is five inches long and represents fifty miles. Divide it into single miles ; also draw a line to represent twenty-eight miles.

(3) On a plan thirty yards is represented by four inches. Draw a scale to show fifty yards, and divide it to read single yards.

(4) In a map one and a quarter inches represent eight miles. Draw a scale to show forty-five miles, and divide it to read single miles.

Problem 17.

To construct a parallelogram having given two adjacent sides and the included angle.



Let e and f be the two given sides, and A the given $\angle$.

CONSTRUCTION.—Draw a line AB $= f$. At A make $\angle$ BAD $= \angle$ A, and make AD $= e$.

With centre D, and radius f, draw an arc of a $\odot$.

With centre B, and radius e, draw an arc cutting the former arc at C. Join CD, CB; join DB.

Then ABCD is the required $\square^{m}$.

PROOF.—In $\triangle$s DAB, DCB,

DC = AB,

DA = CB,

DB is common,

$\therefore \triangle$ DAB $= \triangle$ DCB (I. 14);

$\therefore \angle$ DBA = incl. $\angle$ BDC

$\therefore$ DC $\parallel$ AB (I. 19).

Also DC = AB,

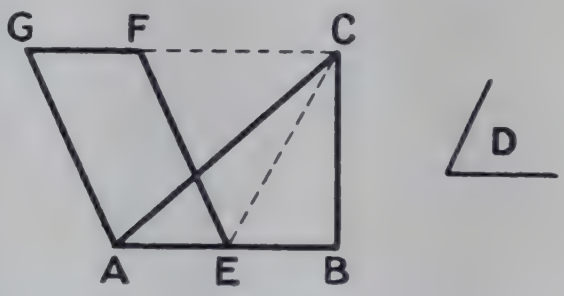
DA = CB,

and DA, CB may also be shown to be $\parallel$,

$\therefore$ ABCD is a $\square^{m}$. Q.E.F

Problem 18.

To construct a parallelogram equal to a given triangle, and having one of its angles equal to a given angle.



Let ABC be the given △, and D the given ∠.

It is required to construct a ▱^{m} equal to the △ ABC, and having one of its ∠s = D.

CONSTRUCTION.—Bisect AB at E.

At E make ∠ AEF = ∠ D.

Through C draw CFG ‖ to AB, and through A draw AG ‖ to EF.

Then ▱^{m} GAEF = △ ABC in area.

PROOF.—Join EC.

Now, △ AEC = △ BEC (I. 28),

∴ △ ABC = 2△ AEC.

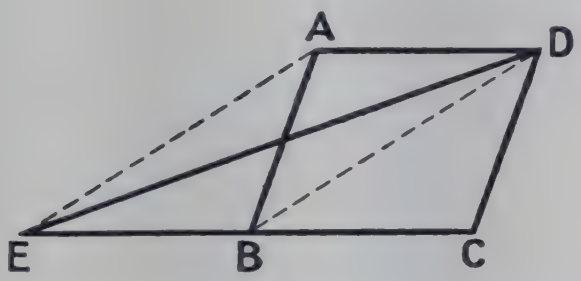
Again, ▱^{m} GAEF = 2△ AEC (I. 28, Cor.);

∴ ▱^{m} GAEF = △ ABC;

and one of its ∠s, ∠ AEF = given ∠ D. Q.E.F.

Problem 19.

To construct a triangle equal in area to a given quadrilateral figure.



Let ABCD be the given quadrilateral.

It is required to construct a △ equal in area to fig. ABCD.

CONSTRUCTION.—Join BD, and draw AE || to BD to meet CB produced in E.

Join ED.

Then EDC is the required △.

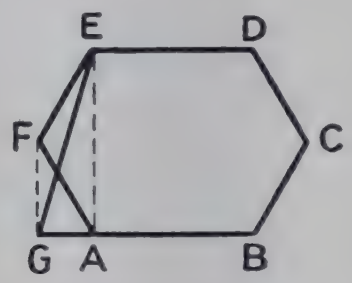
PROOF.—△ EBD = △ ABD (I. 28, Cor. 2);

add △ BCD to each,

then △ EDC = quadrilateral ABCD. Q.E.F.

Problem 20.

To construct a rectilineal figure equal in area to a given rectilineal figure, with one side fewer.



Let ABCDEF be the given rectilineal figure.

It is required to construct a rectilineal fig. equal in area with one side less than the fig. ABCDEF.

CONSTRUCTION.—Join AE.

Draw FG || to EA to meet BA produced in G.

Join GE.

Then fig. GEDCB = fig. ABCDEF in area.

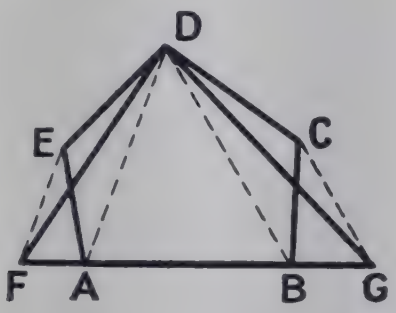
PROOF.—△ GAE = △ FAE (I. 28, Cor. 2),

add ABCDE to each,

∴ fig. GEDCB = fig. ABCDEF. Q.E.F.

Problem 21.

To construct a triangle equal in area to a rectilineal figure of five sides (pentagon).



Let ABCDE be the given irregular pentagon.

It is required to construct a △ equal in area to the fig. ABCDE.

CONSTRUCTION.—Join AD, and through E draw EF || to AD to meet BA produced in F.

Join BD, and through C draw CG || to DB to meet AB produced in G.

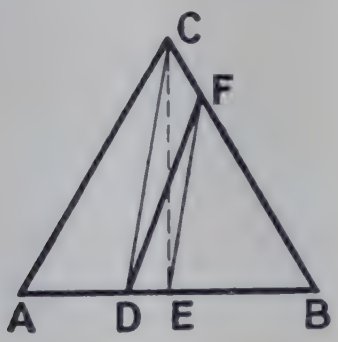
Join FD, GD.

Then FDG is the required △.

Give the proof.

Problem 22.

To bisect a triangle by a straight line drawn through a given point in one of its sides.



Let ABC be the given △, and D the given point in the side AB.

It is required to bisect the △ by a straight line drawn through D.

CONSTRUCTION.—Bisect AB at E ; join CD, CE.

Through E draw EF || to CD.

Join DF.

Then DF bisects the △ ABC.

Supply the proof.

BOOK II

Identity.

Practical Exercises introducing Theorem 1.

(1) Give a geometrical illustration of the product—

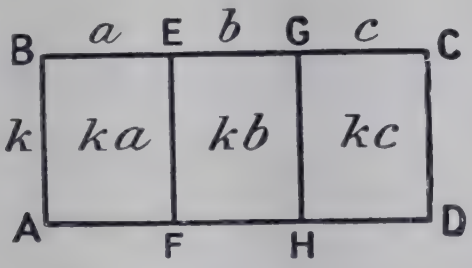
$$a \,.\, b$$

a and b representing units of lengths.

(2) Draw the figure illustrating the algebraical identity, and give explanation, a, b, c, and k being represented by lines—

$$k(a + b + c) = ka + kb + kc$$

Mark the area of each rectangle



(3) Give a geometrical illustration of the following identity, with explanation :—

$$k(a + b + c + d) = ka + kb + kc + kd$$

(4) Illustrate by geometrical construction the identity—

$$k(a - b) = ka - kb$$

(5) Draw a diagram showing the algebraical identity—

$$ab = ba$$

(6) Give an illustration showing geometrically the identity—

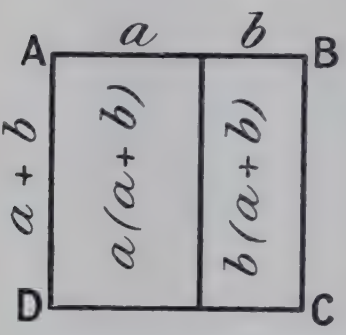
$$(a + b)(c + d) = ac + bc + ad + bd$$

Practical Exercises introducing Theorem 2.

(1) Give a geometrical illustration of the algebraical identity, and give explanation, a and b being lengths—

$$(a + b)^2 = a(a + b) + b(a + b)$$

Indicate the area of each rectangle.



(2) Draw a figure illustrating the algebraical identity—

$$(a+b+c)^2 = a(a + b + c) + b(a + b+c) + c(a + b + c)$$

(3) Illustrate the identity—

$$(2a)^2 = 4a^2$$

(4) Illustrate the identity, a and 4 representing so many units of length—

$$(a + 4)^2 = a(a + 4) + 4(a + 4)$$

(5) Draw a figure illustrating the algebraical identity—

$$(x + 4)^2 = x^2 + 8x + 16$$

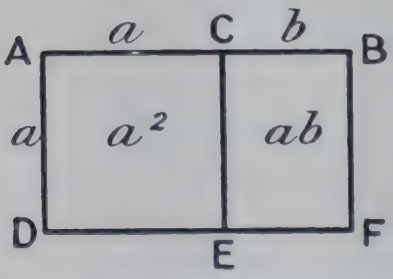
(6) Illustrate the identity—

$$(5 + a)^2 = 25 + 10a + a^2$$

Practical Exercises introducing Theorem 3.

(1) Give a geometrical illustration of the following identity, a and b representing lengths :—

$$a(a + b) = a^2 + ab$$



(2) By means of a figure illustrate the identity—

$$a(a + 3) = a^2 + 3a$$

(3) Draw a geometrical illustration of the identity—

$$a(a + b + c) = a^2 + ab + ac$$

(4) By figure illustrate the identity, x and 6 representing units of lengths—

$$7(x + 5) = 7x + 35$$

(5) Demonstrate the identity—

$$x(a + 4) = ax + 4x$$

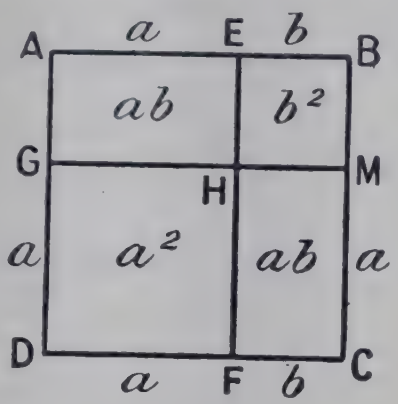
(6) Illustrate the identity—

$$(x + 4)(x + 3) = x^2 + 7x + 12$$

Practical Exercises introducing Theorem 4.

(1) Give a geometrical illustration with explanation of the identity—

$$(a + b)^2 = a^2 + 2ab + b^2$$



(2) Geometrically illustrate the following identity :—

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2bc + 2ca + 2ab$$

(3) Prove by figure the identity—

$$(3a)^2 = 9a^2$$

(4) Demonstrate by means of a figure the identity—

$$(a + b)(b + c) = ac + b(a + b + c)$$

(5) Illustrate geometrically the following identity :—

$$(y + 6)^2 = y^2 + 12y + 36$$

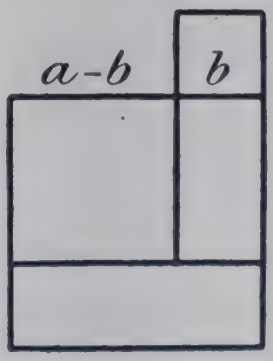
(6) Show by figure the identity—

$$(4 + x)^2 = 16 + 8x + x^2$$

Practical Exercises introducing Theorem 5.

(1) Illustrate by means of a figure giving the lengths of the sides and areas of the contained figures, the identity—

$$(a - b)^2 = a^2 - 2ab + b^2$$



(2) Give a geometrical illustration of the algebraical identity—

$$(x - y)(x - y) = x^2 - 2xy + y^2$$

(3) Illustrate by geometrical construction the identity—

$$(x - 4)^2 = x^2 - 8x + 16$$

(4) Illustrate geometrically the identity—

$$(6 - d)(6 - d) = 36 - 12d + d^2$$

Illustrate the following identities :—

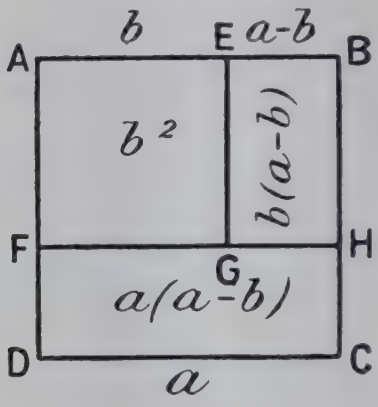
(5) $(y - 7)(y - 7) = y^2 - 14y + 49$

(6) $(x - 4)(x - 3) = x^2 - 7x + 12$

Practical Exercises introducing Theorem 6.

(1) Illustrate by means of a figure the identity—

$$(a^2 - b^2) = (a + b)(a - b)$$



(2) Give a geometrical illustration of the identity—

$$x^2 - 3^2 = (x + 3)(x - 3)$$

(3) Illustrate by a geometrical figure the identity—

$$(x + b)(x - b) = x^2 - b^2$$

(4) Illustrate by a figure the identity—

$$y^2 - 25 = (y - 5)(y + 5)$$

Geometrically illustrate the following identities :—

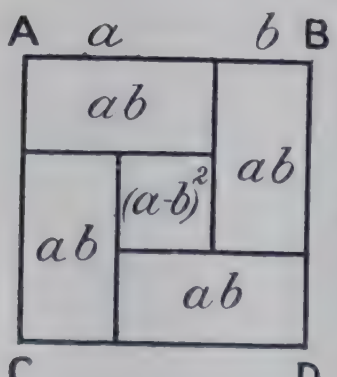
(5) $x^2 - 16 = (x + 4)(x - 4)$

(6) $9^2 - y^2 = (3 - y)(3 + y)$

Practical Exercises introducing Theorem 7.

(1) Illustrate graphically the algebraical identity—

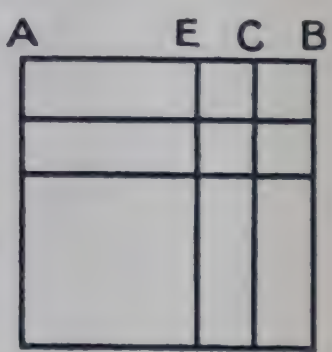
$$(a + b)^2 - (a - b)^2 = 4ab$$



Draw the square AD, and find the areas of the four rectangles and the small square.

(2) Give a geometrical illustration of the algebraical identity—

$$(c + d)^2 = (c - d)^2 + 4cd$$



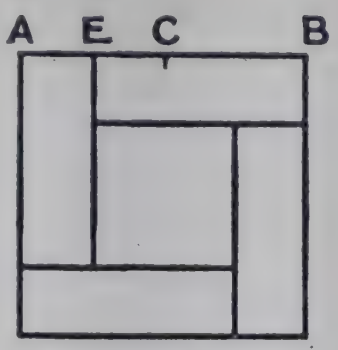
(3) Prove the geometrical illustration of the algebraical identity $(a + b)^2 - (a - b)^2 = 4ab$ from the given figure—

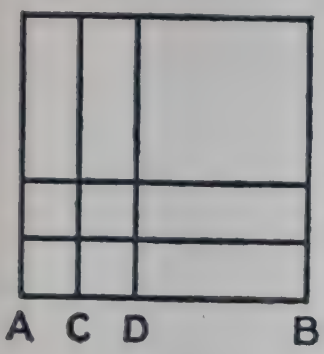
$$AC = a,\ EC = CB = b$$

(4) Repeat the above exercise, drawing a similar figure.

(5) What algebraical identity is suggested by the given figure?—

AC = CB = a, CE = b



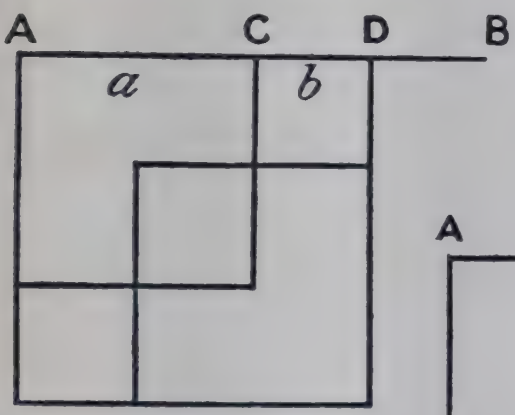


(6) Give the algebraical identity suggested by the following figure :—

CB = a, AC = CD = b

Express your conclusion in words.

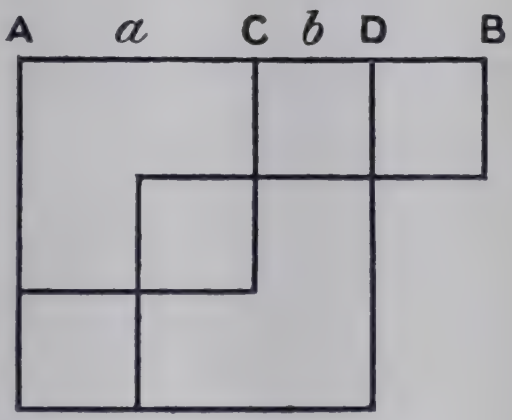
Practical Exercises introducing Theorem 8.



(1) Prove geometrically the algebraical identity—

$(a + b)^2 + (a - b)^2 = 2a^2 + 2b^2$

AC = CB = a, CD = b



(2) From the given figure prove the algebraical identity—

$(a + b)^2 + (a - b)^2 = 2a^2 + 2b^2$

AC = CB = a, CD = b.

(3) Draw a figure illustrating the algebraical identity—

$$2(c^2 + d^2) = (c - d)^2 + (c + d)^2$$

(4) Give a geometrical illustration of the identity—

$$(a + b)^2 + (b - a)^2 = 2a^2 + 2b^2$$

(5) Prove by figure the algebraical identity—

$$(x + y)^2 = 2x^2 + 2y^2 - (x - y)^2$$

(6) Give a geometrical illustration of the algebraical identity—

$$(c + d)^2 + (c - d)^2 = 2(c^2 + d^2)$$

Express in words the conclusion you have arrived at in the foregoing exercises.

Segments of a Line.

In geometry by a straight line we always mean the indefinitely prolonged line, unless the context shows that a finite piece of it is intended.

A finite portion of a straight line is called a segment of it.

Example I.—Let the straight line AB be bisected at C, and divided into unequals at D. AB is said to be divided internally.

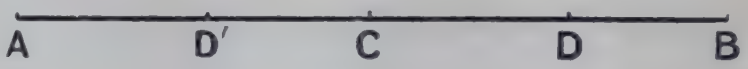


Fig. I.

Example II.—Let the straight line AB be bisected at C, and D a point taken in AB produced. AB is said to be divided externally.

To express the half line AC or CB and the line between the points of section CD in terms of the unequal segments AD, BD—

In Fig. I. $AD + BD = AB = 2AC \text{ or } 2CB,$

$$\therefore AC = CB = \tfrac{1}{2}(AD + DB).$$

In Fig. II. $AD - BD = AB,$

$$\therefore AC = CB = \tfrac{1}{2}(AD - BD).$$

Fig. II.

In Fig. I., if $AD' = BD,$

$$\therefore 2CD = D'D$$
$$= AD - AD'$$
$$= AD - BD,$$
$$CD = \frac{AD - BD}{2}$$

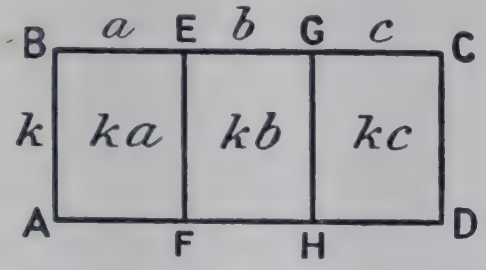
In Fig. II. $$CD = \frac{AD + BD}{2}$$

IDENTITY.

Theorem 1.

Geometrical illustration of the algebraical identity—

$$k(a + b + c) = ka + kb + kc.$$



Draw two straight lines AB, BC at rt. ⊥s; let AB = k units of length, and let BC be divided at E and G into three parts, BE = a units of length, EG = b units of length, GC = c units of length;

$\therefore$ BC = $a + b + c$ units of length.

CONSTRUCTION.—Through the pts. E, G, C draw EF, GH, CD || to BA, each being k units of length.

PROOF.—Rect. AB . BC is made up of rects. AB . BE, FE . EG, HG . GC.

Then rect. AB . BC = rects. AB . BE, FE . EG, HG . GC,

But rect. AB . BC = $k(a + b + c)$ units of area,

And rect. AB . BE = $k \cdot a$ units of area,

" " FE . EG = $k \cdot b$ " "

" " HG . GC = $k \cdot c$ " "

$\therefore k(a + b + c) = ka + kb + kc.$

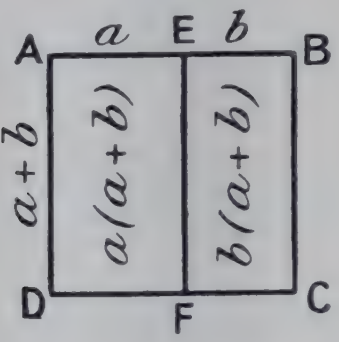
The result may be expressed as follows:—

If there are two straight lines, one of which is divided into any number of parts, the rectangle contained by the two straight lines is equal to the sum of the rectangles contained by the undivided line and the several parts of the divided line.

Theorem 2.

Geometrical illustration of the algebraical identity—

$$(a + b)^2 = a(a + b) + b(a + b).$$



Let the straight line AB be divided at E into two parts, AE = a units of length, EB = b units of length;

$\therefore$ AB = $a + b$ units of length.

CONSTRUCTION.—On AB describe the sq. ABCD; through E draw EF || to AD.

PROOF.—Sq. ABCD is made up of rects. AE . AD, EB . BC;

$\therefore$ sq. ABCD = rects. AE . AD, EB . BC;

But sq. ABCD = $(a + b)^2$ units of area,

and rect. AE . AD = $a(a + b)$ „ „

„ „ EB . BC = $b(a + b)$ „ „

$\therefore (a + b)^2 = a(a + b) + b(a + b)$.

The result may be expressed as follows :—

If a straight line is divided into any two parts, the square on the whole line is equal to the sum of the rectangles contained by the whole line and each of the parts.

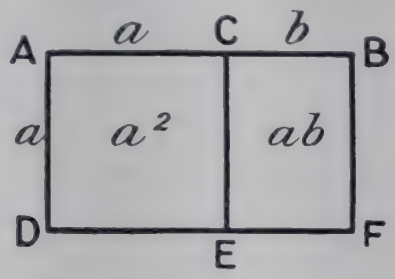
Exercises.

(1) The square on a straight line is equal to four times the square on half the line.

(2) The square on the greater of two straight lines is equal to the rectangle contained by the greater straight line and the difference between the two straight lines with the rectangle contained by the two straight lines.

Theorem 3.

Geometrical illustration of the algebraical identity—
$a(a + b) = a^2 + ab$.



Let the straight line AB be divided into two parts, AC = a units of length, CB = b units of length;

$\therefore$ AB = $(a + b)$ units of length.

CONSTRUCTION.—On AC construct the sq. ACED, and through B draw BF || to AD, meeting DE produced in F.

PROOF.—The fig. AF is made up of figs. AE, CF,

$\therefore$ rect. DA . AB = sq. on AC + rect. FB . BC.

But rect. DA . AB = $a(a + b)$ units of area,

and sq. on AC = a^2 units of area,

rect. FB . BC = ab „ „

$\therefore a(a + b) = a^2 + ab$.

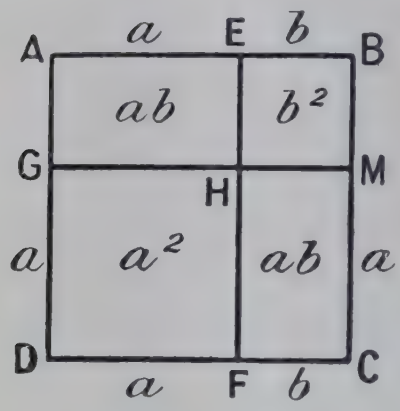
The result may be expressed as follows:—

If a straight line is divided into any two parts, the rectangle contained by the whole line and one of its parts is equal to the square on that part, together with the rectangle contained by the two parts.

Theorem 4.

Geometrical illustration of the algebraical identity—

$$(a + b)^2 = a^2 + 2ab + b^2.$$



Let the strt. line AB be divided into two parts, AE = a units of length, CB = b units of length;

$\therefore$ AB = $(a + b)$ units of length.

Construction.—On AB describe the sq. ABCD; on EB describe the sq. EBHM; produce EH, HM to meet DC in F and AD in G respectively.

Proof.—All the figs. are rectangular, and GF is also a square;

$\therefore$ sq. on AB = sqs. on AE, EB + rects. EA . AG, CM . MH;

But sq. on AB = $(a + b)^2$ units of area,

and sq. on AE = a^2 units of area,

„ EB = b^2 „ „

rect. EA . AG = ab „ „

rect. CM . MH = ab „ „

$\therefore (a + b)^2 = a^2 + b^2 + 2ab,$

$= a^2 + 2ab + b^2.$

The result may be expressed as follows:—

If a straight line is divided into any two parts, the square on the whole line is equal to the sum of the squares on the two parts, together with twice the rectangle contained by the two parts.

Exercises.

(1) Any rectangle is half the rectangle contained by the diagonals of the square described on two of its adjacent sides.

(2) The square on a straight line is equal to four times the square on half that line.

IDENTITY.

Theorem 5.

Geometrical illustration of the algebraical identity—

$$(a - b)^2 = a^2 - 2ab + b^2.$$

Let AB = a units of length.

Cut off CB from AB = b units of length.

Then AC = $(a - b)$ units of length.

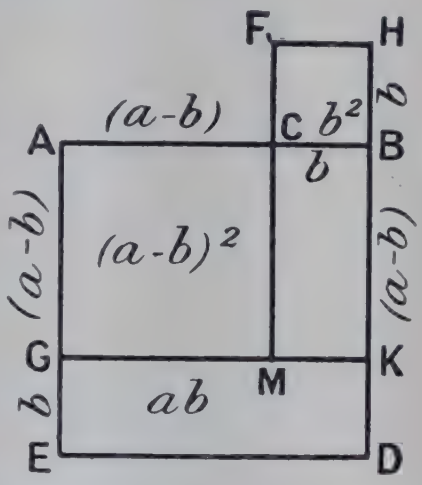
CONSTRUCTION.—On AB draw sq. ABDE;

on CB draw sq. CBHF.

Cut off AG from AE = AC = $(a - b)$ units of length.

Then GE = b units of length.

Through G draw GK || to AB; produce FC to meet GK in M.



PROOF.—Now, the figures are rectangular, and AM, FB are squares.

$\therefore$ sq. on AC = the whole fig. (sq. on AB + sq. on CB) − rect. EK − rect. MH;

But sq. on AC = $(a - b)^2$ units of area

„ AB = a^2 units of area.

„ CB = b^2 „ „

rect. EK = $a \cdot b$ units of area

„ MH = $a \cdot b$ „ „

$$\therefore (a-b)^2 = a^2 + b^2 - 2ab$$
$$= a^2 - 2ab + b^2.$$

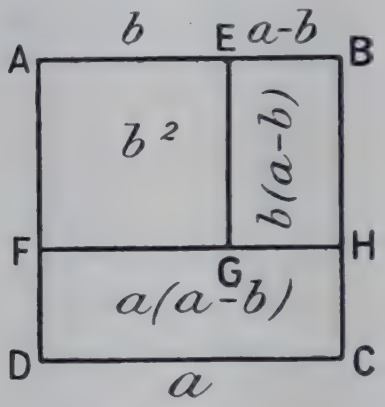
The result may be expressed as follows:—

If a straight line is divided into any two parts, the sum of the squares on the whole line and on one of the parts is equal to twice the rectangle contained by the whole line and that part, together with the square on the other part.

Theorem 6.

Geometrical illustration of the algebraical identity—

$$a^2 - b^2 = (a+b)(a-b).$$



Let AB = a units of length; and on it describe the sq. ABCD.

From AB cut off AE = b units of length,

$\therefore$ EB = $(a-b)$ units of length.

CONSTRUCTION.—From AD cut off AF = AE = b units of length.

Through F draw FH || to DC; through E draw EG || to AD to meet FH in G.

PROOF.—Now, whole fig. AC = fig. AG + fig. DH + fig. EH,

$\therefore$ sq. on AB = sq. on AE + rect. CD . DF + rect. HB . BE;

But sq. on AB = a^2 units of area,

and „ AE = b^2 „ „

rect. CD . DF = $a(a-b)$ units of area

„ HB . BE = $b(a-b)$ „ „

$\therefore a^2 = b^2 + a(a-b) + b(a-b)$;

$\therefore a^2 - b^2 = (a+b)(a-b)$.

The theorem may be expressed as follows:—

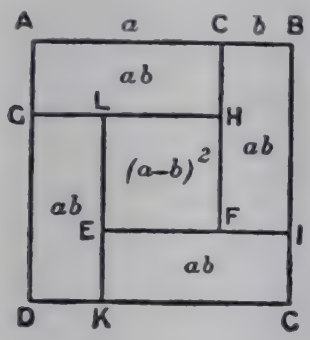
(1) The rectangle contained by the sum and difference of two straight lines is equal to the difference of their squares.

(2) If a straight line is divided equally and also unequally, the rectangle contained by the unequal parts, together with the square on the line between the points of section, is equal to the square on half the line.

Theorem 7.

Geometrical illustration of the identity—

$$(a + b)^2 - (a - b)^2 = 4ab$$



Let the strt. line AB be divided at C, AC = a units of length, CB = b units of length.

$\therefore$ AB = $(a + b)$ units of length.

Construction.—On AB describe the sq. ABCD, and complete the figure as shown in diagram.

Proof.—Sq. on AB = rect. AH + rect. BF + rect. GK + rect. EC + sq. on EF.

But sq. on AB = $(a + b)^2$ units of area,

and ,, EF = $(a - b)^2$,, ,,

rect. AH = $a \cdot b$ units of area,

,, BF = $a \cdot b$,, ,,

,, GK = $a \cdot b$,, ,,

,, EC = $a \cdot b$,, ,,

$\therefore (a + b)^2 = ab + ab + ab + ab + (a - b)^2$;

$\therefore (a + b)^2 - (a - b)^2 = 4ab.$

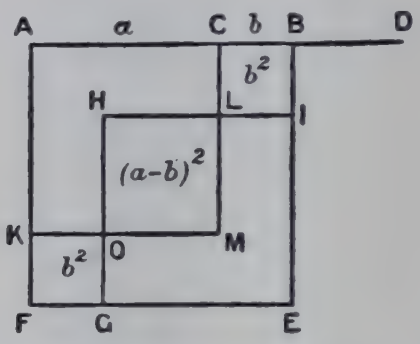
The result may be expressed as follows :—

If a straight line be divided into any two parts, four times the rectangle contained by the whole line and one of the parts, together with the square on the other part, is equal to the square on the straight line which is made up of the whole and that part.

Theorem 8.

Geometrical illustration of the identity—

$$(a + b)^2 + (a - b)^2 = 2a^2 + 2b^2$$



Let the strt. line AD be divided at C and at B, and let AC = CD = a units of length, and CB = b units of length.

$$\therefore \text{AB} = (a + b) \text{ units of length.}$$

Construction.—On AC describe the sq. ACMK.

From EF cut off EG = AC = a units of length; and on EG describe the sq. EGHI.

Proof.—Sq. on AB = sq. on AC + sq. on EG + sq. on CB + sq. on FG − sq. on OM.

But sq. on AB = $(a + b)^2$ units of area,
and ,, AC = a^2 units of area,
,, EG = a^2 ,, ,,
,, CB = b^2 ,, ,,
,, FG = b^2 ,, ,,
,, OM = $(a - b)^2$ units of area;

$$\therefore (a + b)^2 = a^2 + a^2 + b^2 + b^2 - (a - b)^2,$$

$$\therefore (a + b)^2 + (a - b)^2 = 2a^2 + 2b^2. \quad \text{Q.E.D.}$$

The result may be expressed as follows:—

If a straight line be divided equally, and also unequally, the sum of the squares on the two unequal parts is twice the sum of the squares on half the line and on the line between the points of section.

Exercises.

By means of figures illustrate the following identities :—

$$a(b - c) = ab - ac.$$
$$2ab = 2ba.$$
$$(4x)^2 = 16x^2.$$
$$5(a + 6) = 5a + 30.$$
$$7(9 - x) = 63 - 7x.$$
$$(a + b)(c + d) = ac + bc + ad + bd.$$
$$(a + b)^2 - (a - b)^2 = 4ab.$$
$$(a + b)^2 + (a - b)^2 = 2a^2 + 2b^2.$$
$$(x - 4)(x - 6) = x^2 - 10x + 24.$$
$$25 - x^2 = (5 + x)(5 - x).$$

Exercises.

(1) A straight line is divided into two parts : show that if twice the rectangle of the parts is equal to the sum of the squares described on the parts, the straight line is bisected.

(2) Divide a straight line into two parts, such that the rectangle contained by them shall be the greatest possible.

(3) Produce a given line so that the rectangle contained by the whole line thus produced and the part produced is equal to a given square.

(4) Make a rectangle equal to the difference of two given squares.

(5) ABC is a right-angled triangle, having a given angle at C. From C a perpendicular is drawn, meeting AB in D : show that the rectangle AB . BD is equal to the square on BC. Similarly, show BA . AD equals the square on AC.

(6) In a right-angled triangle the square on the hypotenuse equals four times the area of the triangle together with the square on the difference of the two sides.

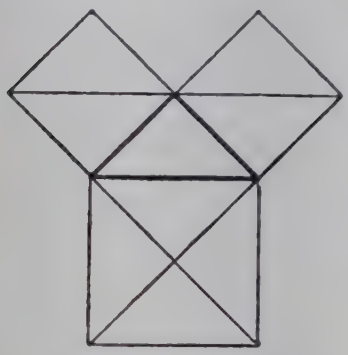
(7) If a perpendicular be drawn from the right angle of a right-angled triangle to the hypotenuse, the square on this perpendicular is equal to the rectangle contained by the segments of the hypotenuse.

(8) Any rectangle is half the rectangle contained by the diagonals of the squares described upon its two sides.

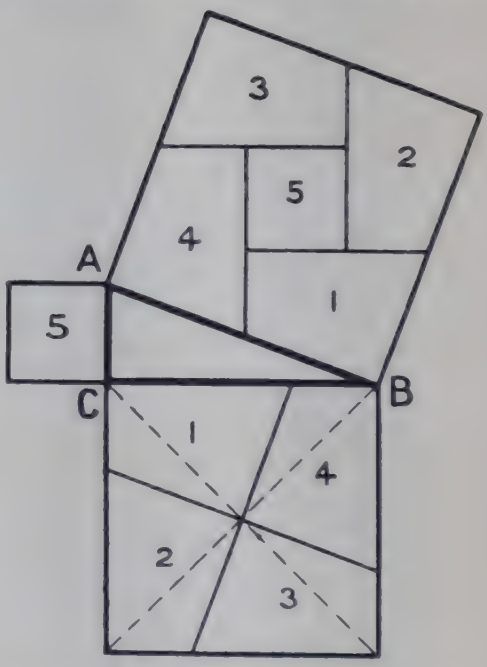
THEOREM OF PYTHAGORAS.

Practical Exercises introducing Theorem 9.

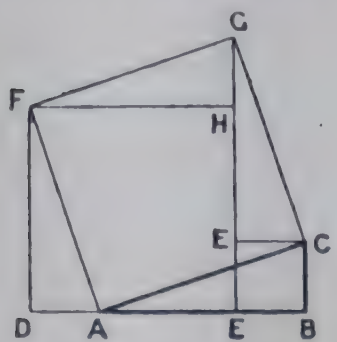
(1) Construct a right-angled triangle with sides of 3 units, 4 units, and 5 units length respectively; upon each side describe a square. Divide each square into square units of area, and find out whether the square units in the square on the hypotenuse is equal to the sum of the square units of the squares on the other two sides.



(2) Construct an isosceles right-angled triangle having its equal sides 2″ long. Upon each side describe a square, and join the opposite corners as in figure. Compare the areas of the triangles formed, and the sum of the areas of the squares on the two equal sides with the area of the square on the hypotenuse.



(3) Draw a large-sized right-angled triangle ABC on a sheet of paper and draw squares on AC, CB, and the hypotenuse AB. Find the mid-point of the square on CB by drawing the diagonals. Through the centre draw two lines—parallel and perpendicular to AB respectively. Cut out the four pieces marked 1, 2, 3, 4. By superposition it will be found that the four pieces in this square, together with the square on AC, exactly fit into the square on AB in the way shown by the corresponding number.

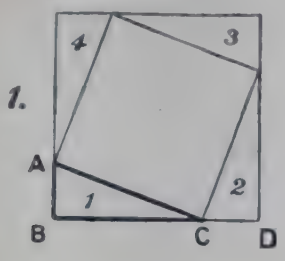


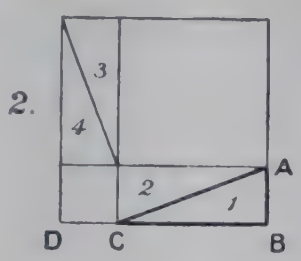
(4) (*a*) In the opposite figure, given—

$$AB = DE = DF$$
$$BC = DA$$

prove that $AC^2 = AB^2 + BC^2$.

(*b*) From the figures (I) and (II) prove that $AC^2 = BC^2 + AB^2$, given CD = AB.





(5) Make a right-angled triangle ABC with sides of 8 cm. and 6 cm. containing the right angle. Find the length of the hypotenuse and square it. Is the square on the hypotenuse equal to the sum of the squares on the other two sides?

(6) Repeat the last exercise, making the sides containing the right angle—

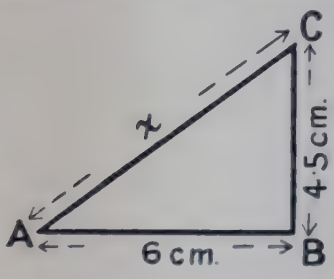
AB = 5 cm., BC = 12 cm.
AB = 4·5 cm., BC = 6 cm.

What do you learn from the above experiments?

Practical Exercises introducing Theorem 10.

(1) Construct a triangle ABC, given AB = 3 cm., BC = 4 cm., CA = 5 cm. Measure the angle ABC. What kind of an angle is it?

(2) Square the lengths 8 cm. and 6 cm.; find the sum of their squares, and extract the square root of their sum. Take the square root as the length of the third side of a triangle, and construct the triangle. Measure the angle formed by the sides 8 cm. and 6 cm. How many degrees does it contain?

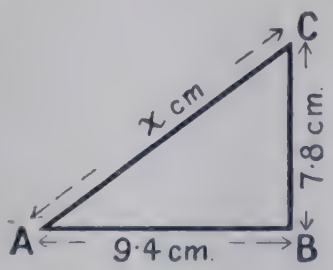


(3) Square the length of the sides AB, AC, and get the square root of the sum of their squares. Let this be the length of the hypotenuse of the triangle ABC. Make the triangle, and find the value of the angle ABC.

(4) Repeat Exercise 3, substituting the values AC = 9·4, CB = 7·8 cm. What do you learn from the above exercises?

Determine whether or not the following triangles are right-angled :—

14 cm., 48 cm., 50 cm.
5 cm., 12 cm, 13 cm.

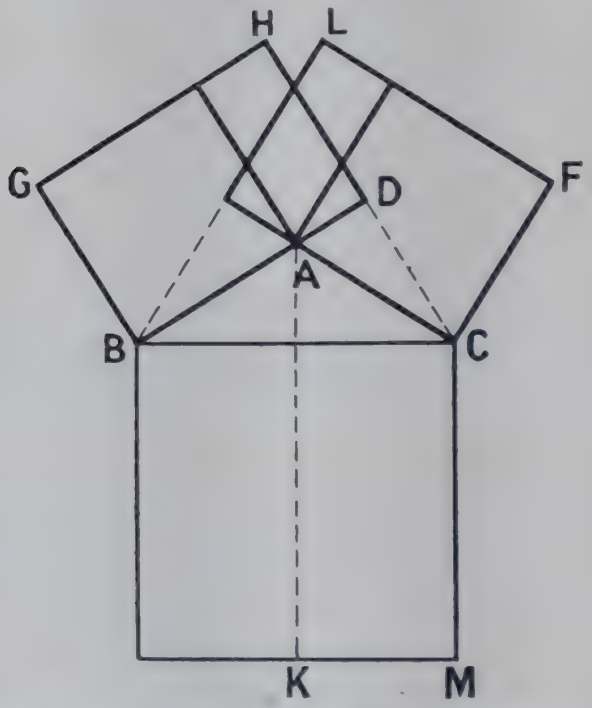


Squares on Sides of a Triangle.

Practical Exercises introducing Theorem 11.

(1) Construct any large obtuse-angled triangle ABC. On the three sides draw squares, and complete the figure as shown in diagram. By measurement and calculation prove that in area—

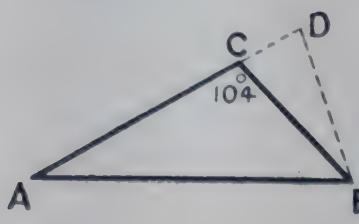
(*a*) BK = GA + HA.
(*b*) CK = FA + LA.
(*c*) $BC^2 = BA^2 + AC^2 + HA + LA$
$= BA^2 + AC^2 + 2BA \cdot AD.$



(2) Confirm the above results by another experiment.

(3) By calculation prove that—

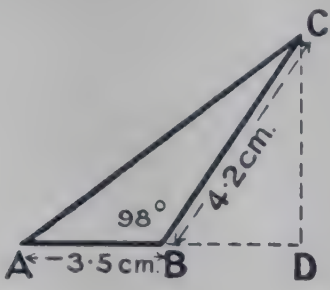
$$AB^2 = AC^2 + CB^2 + 2AC \cdot CD.$$



(4) Repeat Exercise 3, but taking a different obtuse-angled triangle.

(5) Prove arithmetically that—

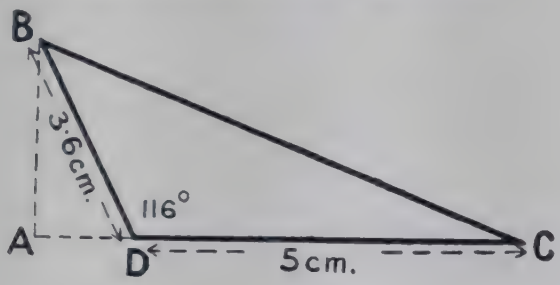
$$AC^2 = AB^2 + BC^2 + 2AB \,.\, BD.$$



(6) Arithmetically prove that—

$$BC^2 = DC^2 + BD^2 + 2CD \,.\, DA.$$

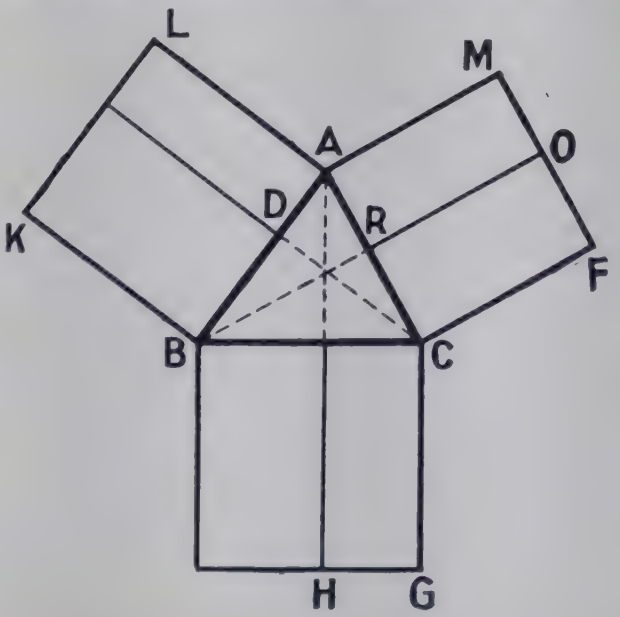
What do you conclude from the above exercises?



Practical Exercises introducing Theorem 12.

(1) Construct any large acute-angled triangle ABC. On the three sides describe squares, and complete the construction as in figure. By measurement and calculation show that in area—

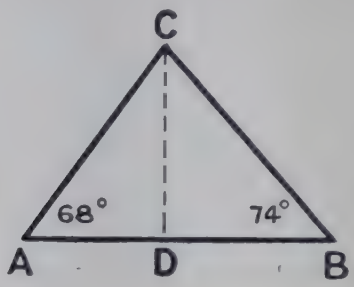
(*a*) BH = KD.
(*b*) CH = FR.
(*c*) $BC^2 = AB^2 + AC^2 - LD - MR$
$= AB^2 + AC^2 - 2BA \,.\, AD.$



(2) Repeat the above exercise, taking another acute-angled triangle to confirm the last results.

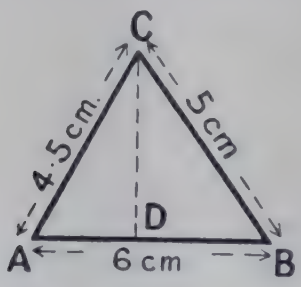
(3) By calculation prove that—

$$AC^2 = AB^2 + BC^2 - 2AB \cdot BD.$$



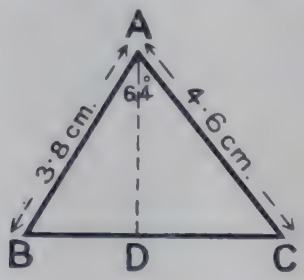
(4) Prove arithmetically that—

$$AC^2 = AB^2 + BC^2 - 2BC \cdot CD.$$



(5) Prove arithmetically that—

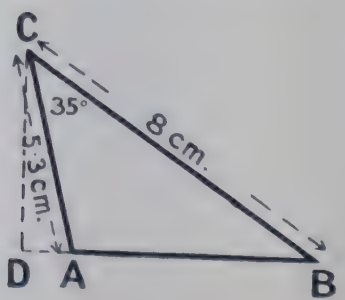
$$AB^2 = AC^2 + BC^2 - 2BC \cdot CD.$$



(6) Show that—

$$AC^2 = AB + BC^2 - 2AB \cdot AD.$$

What do you gather from these exercises?



PYTHAGORAS' THEOREM.

Theorem 9.

In a right-angled triangle the square described on the hypotenuse is equal to the sum of the squares described on the other two sides.

Let ABC be a rt.-angled △, having the ∠ BAC a rt. ∠.

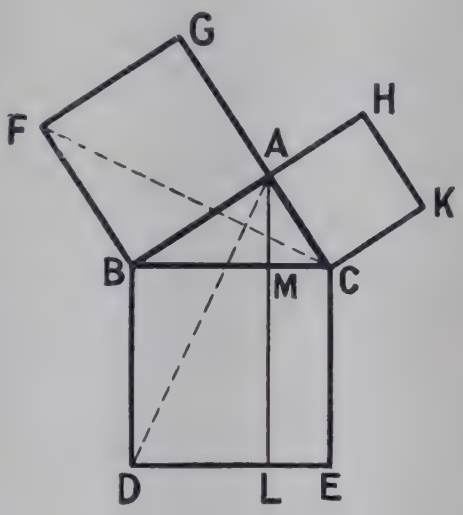
Upon AC, BA, BC let the sqs. AK, AF, BE be respectively described.

Then

sq. BE = sq. AF + sq. AK.

CONSTRUCTION.—Through A draw AL || to BD.

Join CF, AD.



PROOF.—

∵ adj. ∠ s CAB, BAG are rt. ∠ s,

∴ CA, AG are in the same straight line (I. 2).

Sim$^{ly.}$ BA, AH are in the same straight line.

Now, ∠ FBA = ∠ CBD,

To each add ∠ ABC,

∴ ∠ FBC = ∠ ABD.

Hence in △s ABD, FBC,

AB = FB,

BD = BC,

∠ ABD = ∠ FBC,

∴ △s ABD, FBC are congruent (I. 7).

Now, the sq. AF is double of the △ FBC,

∵ they are on the same base and of the same altitude (I. 28).

Also the ▱^{m} BL is double of the △ ABD (Cor. 3, I. 28),

∴ ▱^{m} BL = sq. AF;

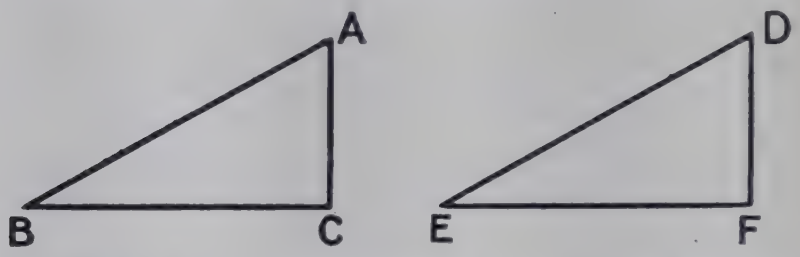
Sim$^{ly.}$ ▱^{m} CL = sq. AK,

∴ the whole sq. BE = sq. AF + sq. AK. Q.E.D.

(This theorem is known as the **Theorem of Pythagoras.**)

Theorem 10.

If the square described on one side of a triangle is equal to the sum of the squares described on the other sides, then the angle contained by these two sides is a right angle.



Let ABC be a △, and square on AB = sum of squares on BC, CA.

Then ∠ ACB = rt. ∠.

CONSTRUCTION.—Make a △ DEF in which DFE = rt. ∠, DF = AC, and EF = BC.

PROOF.— ∵ ∠ DFE = rt. ∠,

∴ sq. on ED = sq. on EF + sq. on DF (II. 9)

= sq. on BC + sq. on AC (Const.)

= sq. on AB (Hyp.);

∴ ED = AB;

∴ in △s ABC, DEF,

AB = DE,

BC = EF,

AC = DF,

∴ △s ABC, DEF are congruent (I. 14);

∴ ∠ ACB = ∠ DFE.

But ∠ DFE = rt. ∠ (Const.);

∴ ∠ ACB = rt. ∠. Q.E.D.

Exercises.

(1) Describe a square equal to the sum of three or more squares.

(2) Describe a square equal to the difference of two squares.

(3) Divide a straight line into two parts, so that the square on one part is double of the square on the other.

(4) The square on the side subtending an acute angle of a triangle is less than the squares on the sides containing the acute angle.

(5) The square on the side subtending an obtuse angle of a triangle is greater than the squares on the sides containing the obtuse angle.

(6) Squares standing on equal straight lines are equal.

(7) If any point P be joined to A, B, C, D, the angular points of a rectangle, the squares on PA and PC are together equal to the squares on PB and PD.

(8) Squares which are equal stand on equal bases.

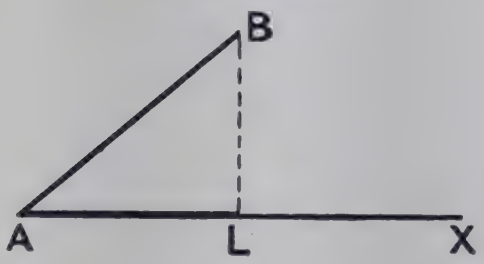
(9) Prove that the square on the diagonal of a given square is double that of the given square.

(10) Divide a straight line into two parts, such that the sum of their squares shall be equal to a given square.

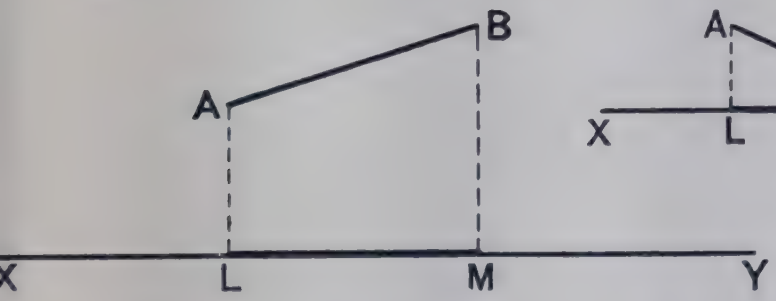
(11) Determine whether or no the following triangles are right-angled :—

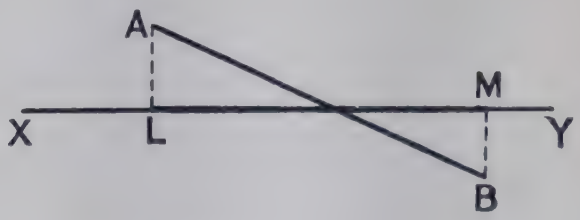
4·5 cm., 6 cm., 7·5 cm.
5 cm., 12 cm., 13 cm.
14 cm., 48 cm., 50 cm.

Projections.



If AX is an indefinite straight line, if AB is a finite straight line, and BL is a perpendicular, then AL is called the **projection** of AB on AX.



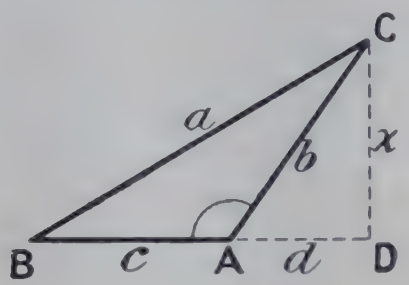


If AL, BM are perpendiculars to XY, LM is called the projection of AB on XY.

Squares on Sides of a Triangle.

Theorem 11.

In an obtuse-angled triangle, the square on the side opposite the obtuse angle is greater than the sum of the squares on the sides containing the right angle by twice the rectangle contained by one of those sides and the projection on it of the other.



Let the △ ABC have ∠ BAC obtuse; let CD be the ⊥ from C to meet BA produced at D;

∴ AD = projection of AC upon BA.

Let BC = a, BA = c, AC = b, AD = d, CD = x.

Then $BC^2 = CA^2 + BA^2 + 2BA \cdot AD$.

Proof.— ∵ BDC is a rt.-angled △,

∴ $BC^2 = BD^2 + DC^2$ (II. 9),

∴ $a^2 = (c + d)^2 + x^2$

$= c^2 + 2cd + d^2 + x^2$.

∵ ADC is a rt.-angled △,

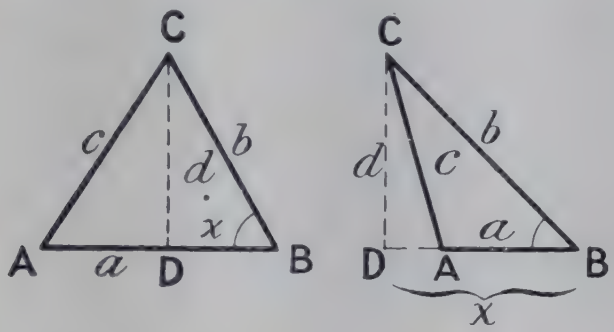
∴ $b^2 = d^2 + x^2$;

∴ $a^2 = b^2 + c^2 + 2cd$;

∴ $BC^2 = CA^2 + BA^2 + 2BA \cdot AD$. Q.E.D.

Theorem 12.

In any triangle, the square on the side opposite an acute angle is less than the sum of the squares on the sides containing the acute angle, by twice the rectangle contained by one of those sides and the projection on it by the other.



Let $\triangle$ ABC have $\angle$ ABC acute; let CD be the $\perp$ from C upon AB, or AB produced,

$\therefore$ BD = projection of CB upon AB.

Let AB = a, BC = b, CA = c, DB = x, CD = d,

Then $AC^2 = AB^2 + BC^2 - 2AB \cdot BD$.

PROOF.— $\because$ ADC is a rt.-angled $\triangle$,

$AC^2 = AD^2 + DC^2$ (II. 9);

In fig. 1 $\therefore c^2 = (a - x)^2 + d^2$;

In fig. 2 $\therefore c^2 = (x - a)^2 + d^2$;

$\therefore$ in the two figures

$c^2 = a^2 - 2ax + x^2 + d^2$.

$\because$ BDC is a rt.-angled $\triangle$,

$\therefore b^2 = x^2 + d^2$;

$\therefore c^2 = a^2 - 2ax + b^2$,

$\therefore AC^2 = AB^2 + BC^2 - 2AB \cdot BD$.

Q.E.D.

Exercises.

(1) Describe an isosceles triangle, having the square on the base equal to three times the square on either of the equal sides.

(2) The square on the base of an isosceles triangle is equal to twice the rectangle contained by either side, and by the straight line intercepted between the perpendicular let fall on the side from the opposite angle and the extremity of the base.

(3) The sum of the squares on the sides of a parallelogram is equal to the sum of the squares on the diagonals.

(4) In any quadrilateral the squares on the diagonals are together equal to twice the sum of the squares on the straight lines joining the middle points of opposite sides.

(5) A square BDEC is described on the hypotenuse BC of a right-angled triangle ABC. Show that the squares on DA and AC are together equal to the squares on EA and AB.

(6) Prove that three times the sum of the squares on the sides of a triangle is equal to four times the sum of the squares on the medians of the triangle.

(7) A side AB of an isosceles triangle ABC is produced to D, so that ABD = AB. Show that $BD^2 = 2BC^2 + AB^2$.

(8) The sum of the squares on any two sides of a triangle equal twice the sum of the squares on half the base, and on the line joining the vertical angle with the middle point of the base.

Exercises on Book II

(1) The square of the perpendicular upon the hypotenuse of a right-angled triangle from the right angle is equal to the rectangle under the segments of the hypotenuse.

(2) The sum of the squares of the diagonals of any parallelogram is equal to the sum of the squares of the sides.

(3) If perpendiculars are drawn from the three angles of any triangle to the opposite sides, or sides produced, the sum of the squares on the alternate segments of the sides are equal.

(4) Divide a given straight line into two such parts that the rectangle contained by them shall be greatest possible.

(5) Show that the square on the sum of two straight lines together with the square on their difference are double the squares on the two straight lines.

(6) ABC is a triangle in which B is a right angle, and DE is drawn from a point D in AB perpendicular to AC. Prove that the rectangle CA . AE equals the rectangle BA . AD.

(7) Divide a given straight line into two such parts that the sum of their squares shall be equal to a given square.

(8) The square on the side subtending an obtuse angle of a

triangle is greater than the squares on the sides containing the obtuse angle.

(9) If a straight line AB be divided in medial section in the point C, then $AB^2 + BC^2 = 3AC^2$.

(10) Construct a rectangle equal to the difference of two squares.

(11) The sum of the squares on the sides of a parallelogram is equal to the sum of the squares on the diagonals.

(12) To produce a given straight line so that the rectangle contained by the former and another given line shall be equal to the square on the part produced.

(13) Produce a given line so that the rectangle of the whole line produced and the original line shall be equal to a given square.

(14) Construct a rectangle equal to a given square when the sum of two adjacent sides of the rectangle is equal to a given line.

(15) Find the side of a square equal in area to a given equilateral triangle.

(16) Any rectangle is equal to half the rectangle contained by the diagonals of the squares upon its sides.

(17) Find the side of a square equal in area to a given equilateral triangle.

(18) ABC is a right-angled triangle having a right angle at C. From C a perpendicular is drawn meeting AB in D. Show that the rectangle AB . BD is equal to the square on BC. Similarly prove that BA . AD equals the square on AC.

(19) Divide a straight line into two parts so that the square on one part is double of the square on the other.

(20) In a triangle whose vertical angle is a right angle a straight line is drawn from the vertex perpendicular to the base. Show that the square on either of the sides adjacent to the right angle is equal to the rectangle contained by the base and the segment of it adjacent to that side.

(21) ABC is an equilateral triangle, and AD is the perpendicular drawn from A to BC. Show that the square on AD is three times the square on BX.

(22) Describe a square equal to the sum of two squares.

(23) Any rectangle is half the rectangle contained by the diagonals of the squares described upon its two sides.

(24) If a straight line DE be drawn parallel to the base BC of an isosceles triangle ABC, and the points B, E be joined, the square on BE shall be equal to the rectangle contained by BC and DE together with the square on CE.

BOOK III.

The Circle.

The circle is the most perfect of plane figures; it is also the simplest form of a curved line.

A circle is a plane figure contained by one line, which is called the circumference, and is such that all straight lines drawn from a certain point within the figure to the circumference are equal to one another. This point is the **centre** of the circle.

A **radius** of a circle is a straight line drawn from the centre to the circumference.

A **diameter** of a circle is a straight line drawn through the centre and terminated both ways by the circumference.

A **semicircle** is the figure bounded by a diameter of a circle and part of the circumference cut off by the diameter on one side.

A **segment of a circle** is the figure bounded by any straight line and part of the circumference which it cuts off.

A **chord** of a circle is a straight line joining any two points on the circumference.

An **arc** of a circle is any part of the circumference.

A chord divides the circumference into two arcs. If the arcs be unequal, the greater one is called the major arc, the less is the minor arc.

A **secant** is a chord produced both ways.

Practical Exercises on the Circle.

(1) From a given point A describe a circle. Indicate—
  (*a*) the centre,
  (*b*) the circumference.

(2) Describe any circle ABC.
  (*a*) Through the centre draw a straight line terminating both ways by the circumference (diameter).

(*b*) From the centre draw a straight line to the circumference (radius).

(*c*) Measure the lengths of the diameter and radius: compare the results. State your conclusion.

(3) Describe a circle, and draw its diameter. How is the circle divided? Use tracing-paper to prove your answer.

(4) Describe a circle of radius 1·5 in.; draw its diameter and several other parallel straight lines. Measure the length of the lines and tabulate them in order of magnitude. What do you observe?

(5) Draw a circle ABC having a radius of 2 in. In it draw a chord AB. Show by shading which is a segment of the circle.

(6) Describe any circle showing a chord, a segment, a sector, arcs major and minor.

Chord and Centre.

Practical Exercises introducing Theorem 1.

(1) In a circle draw a chord; bisect the chord; join the centre of the circle to the mid-point of the chord. Measure the adjacent angles so formed. What do you observe?

(2) In a circle draw several chords; bisect them, and join their mid-points to the centre of the circle. What do you notice? State your conclusion.

(3) In a circle draw a chord. From the centre of the circle draw a perpendicular to the chord. Measure the segments of the chord.

(4) In a circle draw several chords. From the centre of the circle draw perpendiculars to the chords; measure their segments. What do you conclude?

(5) In a given circle draw any chord. Prove that the line joining the centre of the circle to the mid-point of the chord is perpendicular to the chord.

(6) Two chords of a circle are given in position and magnitude; find the centre of the circle.

To obtain the Centre.

Practical Exercises introducing Theorem 2.

(1) Draw a circle passing through any three points not in a straight line; join the three points and bisect the lines perpendicularly by straight lines. The point where the bisectors meet is the centre of the circle.

(2) Describe a circle passing through any three points A, B, C, not in a straight line. Which is the locus of all the points equidistant from AB, BC respectively? Which point, therefore, is equidistant from A, B, C? Can any other point be equidistant from A, B, C? How many circles can pass through A, B, C? Write down your observations.

(3) Draw any circle ABC; take D any point without the circle. With D as centre, and taking different radii, draw a number of circles cutting the circle ABC. In how many points does each circle intersect the circle ABC?

(4) Describe a circle about the triangle ABC, AB = 2·9 in., AC = 2·5 in., CB = 1·8 in.

(5) Describe a circle about an isosceles triangle ABC, AB = AC = 5·6 cm., BC = 4·2 cm.

(6) Describe a circle which shall pass through three given points A, B, C, not in a straight line, the point B being 6·4 cm. distant from A, the point C being 4 cm. distant from A, and also 5·6 cm. from B.

Congruence of Circles.

Practical Exercises introducing Theorem 3.

Use of **Compasses** and **Scale** for finding the length of a curved line.

Draw a curved line, AB. Within the legs of the compasses take a small unit of length, the smaller the better, and, starting from A, see how many times that unit of length is contained in the curved line AB. If there be a fraction of the unit over, as CB, take that distance between the compasses and measure it on a scale.

N.B.—This method is only an approximation, as one is measuring a number of chords.

(1) Describe two equal circles of radius not less than 1·5 in. At their centres make angles of 53°. Measure by means of compasses and scale the length of the arcs of the circle. What do you learn?

(2) On tracing paper draw another similar circle to those in Exercise 1, also having an angle of 53° at the centre. By superposition of this circle upon each of the above circles state what you observe about the arcs.

(3) Draw two equal circles; mark off arcs on the circumferences equal to the radius. Measure the angles subtended by these arcs, and compare the magnitudes of the angles.

(4) Make a similar circle to those in Exercise 3 on tracing-paper, and mark off an arc equal to the radius. By superposition of this circle upon each of the above circles compare the values of the angles subtending the arcs. What is your conclusion?

(5) D is a point in the circumference of a circle equidistant from the radii CA, CB. Prove by construction that arc AD equals arc BD.

(6) AB, AC are a chord and diameter meeting at a point A in the circumference of a circle. Show that the radius drawn parallel to AC bisects the arc BC.

Practical Exercises introducing Theorem 4.

(1) Describe two circles; in each make R equal chords AB, CD respectively. Measure the lengths of the arcs, using compasses and scale, and compare their lengths.

(2) On tracing-paper describe a similar circle to one in Exercise 1, and draw a similar chord. Apply the circle on each of the above circles. Compare the major and minor arcs cut off by the chord of the one, with the major and minor arcs cut off by the chord of the other. Give conclusion.

(3) Draw two circles; mark off equal arcs. Measure the length of the chords. What do you notice?

(4) Applying the method described in 2, tell what you learn about the relation between the equal arcs in two equal circles and the chords of the arcs.

(5) In a circle place six chords each equal to the radius; in another circle place twelve chords each equal to half the radius.

(6) Inscribe a quadrilateral ABCD in a circle, making AB equal to CD. Prove that AC equal BD.

Practical Exercises introducing Theorem 5.

(1) In a circle draw various chords at different distances from the centre. Measure their lengths. What do you observe?

(2) Draw a circle; in it draw a few equal chords. Compare their distances from the centre. Do the results agree?

(3) In a circle draw equal pairs of lines from the centre, not touching the circumference of the circle; at right angles to these lines draw chords. Measure the chords. What are your observations?

(4) Calculate arithmetically the length of a chord which stands at a distance of 6 in. from the centre of a circle radius 1·4 in. Test answer by measurement. What is your error of construction?

(5) Find the locus of the mid-points of chords 6 cm. in length in a circle of radius 5 cm.

(6) Find the radius of a circle given that a chord 3 in. long is 2 in. from the centre. Verify result by construction.

Exercises on Theorems 1 to 5.

(1) AB, AC are two equal chords of a circle. Show that the straight line which bisects the angle BAC passes through the centre of the circle.

(2) Two chords of a circle are given in position and magnitude. Find the centre of the circle.

(3) Describe a circle that shall pass through two given points 3 4 in. apart, having a radius of 2 in.

(4) ABC, DBC are two triangles on the same base and on the same side of it. Make AB equal to BD, and prove that AC and CD are equal.

(5) Show by construction that the parts of a straight line intercepted between the circumferences of concentric circles are equal.

(6) Given the arc of a circle, describe the whole circumference of which the given arc is a part.

(7) Bisect the arc AB of any circle in C. Prove that the chord AB is less than twice the chord AC.

(8) Given that a chord 12 cm. long is distant 2·5 cm. from the centre, calculate (1) the length of a chord distant 5 cm. from the centre; (2) the distance from the centre of a chord 6 cm. long. Check your results by measurement.

(9) AB is an arc of a circle bisected in C. Construct a chord AD equal to half the chord AB, and prove by measurement that the arc AD is less than the arc AC.

(10) Draw a chord of a circle—

(*a*) equal and parallel to a given chord.

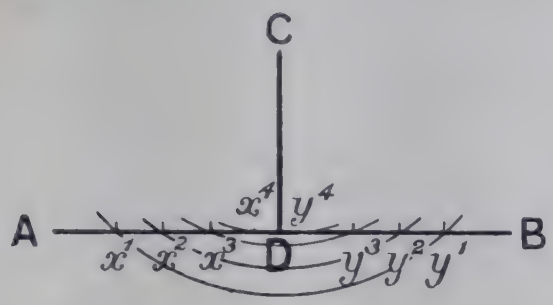
(*b*) equal and perpendicular to a given chord.

Tangency.

Practical Exercises introducing Theorem 6.

(1) Draw a straight line CD, bisecting the straight line AB at right angles in D. With centre C draw a circle, cutting AB in two coincident points x^1 and y^1. Continue the process of drawing circles till CD is used as radius. Compare Dx^1 and Dy^1, Dx^2 and

Dy^2 . . . Dx^4 and Dy^4 . . . How does the last circle meet AB? What do you learn?



(2) Draw a straight line perpendicularly through the point of contact of a tangent to a circle. Where does the line pass?

(3) Draw a large circle and a line touching it by moving a ruler till the chord vanishes. Find by construction the exact point of contact.

(4) In any circle join the centre to the point of contact of a tangent. Measure the angles so formed, and tell what you observe.

(5) Draw a tangent to a circle which is—
(*a*) parallel.
(*b*) perpendicular to a given straight line.

(6) Prove that a chord makes equal angles with the tangents at its extremities.

Practical Exercises introducing Theorem 7.

(1) Take a point A outside a circle whose centre is D; join AD. Describe a circle on AD, cutting the circle at B and C; join AB, AC. What are the magnitudes of the angles ABD, ACD? Compare the angles ADC, ADB and the lengths AB, AC. Give your inferences.

(2) Find a point A outside a circle, such that the tangents from A to the circle may include an angle of 80°.

(3) What is the greatest angle which tangents to a circle can include, and what is the least? Prove by construction.

(4) Find a point D outside a circle, such that the length of the tangent from D to the circle is 1·5 in. Find all the points fulfilling these conditions, and join them.

(5) Two parallel tangents meet a third tangent at E and F. Prove that EF subtends a right angle at the centre.

(6) Describe a circle to touch two given parallel straight lines and a given straight line which intersects them. How many solutions are there?

Contact of Circles.

Practical Exercises introducing Theorem 8.

(1) Draw a circle with A as centre cutting another circle in two coincident points x, y. Continue the process of drawing circles with A as centre till one circle touches the other at x^4y^4; *i.e.* meets it in two coincident points. Indicate the point of contact.

(2) Draw two circles touching externally; join their centres. Prove that the point of contact of the two circles lies in the straight line through the centres.

(3) Describe two circles touching internally. Join their centres, and produce the line both ways through the circles. Prove that the point of contact of the two circles lies in the straight line joining the centres.

(4) Draw four circles of unequal radii each touching two and only two of the others. Show that a circle can be inscribed in the quadrilateral formed by joining the centres.

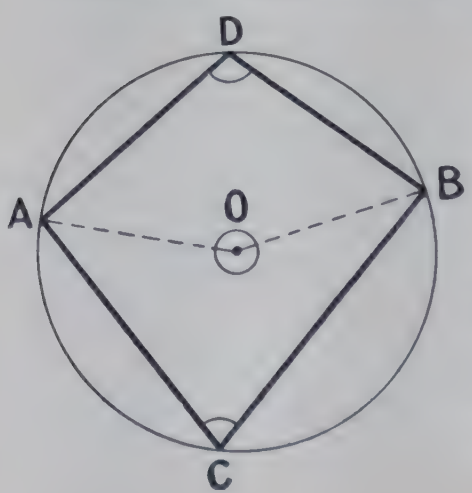
(5) Draw a number of circles of radius 2 in. to touch a circle of radius 2 in., and enclose it.

(6) Describe two circles touching each other externally at A, and touched by a straight line at B and C, and prove that BAC is a right angle.

Angles at Circumference.

Practical Exercises introducing Theorem 9.

Compare the angle which an arc of a circle subtends at the centre with the angle it subtends at any point on the remaining part of the circumference.



(1) Where the centre O is within the angle at the circumference.

(2) Where the centre O is without the angle at the circumference.

(3) Find and compare the magnitudes of the marked angles in a figure similar to the given one.

Also give—

(*a*) the sum of the angles at the centre.
(*b*) the sum of the angles CAD, ADB.
(*c*) the sum of the angles DBC, BCA.

(4) Construct an isosceles triangle having a base of 2·4 in. and a vertical angle of 70°. Where would you look for the vertex of another triangle on the same base but having a vertical angle of 35°?

(5) Draw a circle and a diameter, and on the circumference of the circle mark any four points. Measure the angle subtended by the diameter at each of these points.

(6) Repeat exercise 4, substituting angles 40° and 80° for angles 70° and 35° respectively.

Angles in a Segment.

Practical Exercises introducing Theorem 10.

(1) Draw a circle; in the same segment which is made greater than a semicircle make two angles. Measure and compare the magnitudes of these angles at the circumference. What is observed?

Compare their magnitudes with the angle which is subtended at the centre.

(2) Repeat the above exercise with the segment equal to a semicircle.

(3) Repeat exercise 1, with the segment less than a semicircle. Give observations.

(4) AB is the chord of the segment of a circle. C is any point on the arc of the segment. Prove that the sum of the angles CAB, CBA is the same for every position.

(5) Draw two chords AB, CD of a circle intersecting at E; prove that the triangles PED, CEB are equiangular.

(6) D is any point on the arc of a segment of which AB is the chord. Prove that the sum of the angles DAB, DBA is constant.

Practical Exercises introducing Theorem 11.

(1) Draw a circle ABC; make an angle—
(*a*) in a semicircle.
(*b*) in a segment greater than a semicircle.
(*c*) in a segment less than a semicircle.

Give the magnitudes of the angles and state whether the angles are equal to, greater than, or less than a right angle.

(2) Show that a circle described on the hypotenuse of a

right-angled triangle as diameter passes through the opposite angular point.

(3) A, B, C, D are points on a circle; the diagonals meet at E. Prove that the triangles ABE, DCE are equiangular, and also that the triangles BED, AEC are equiangular.

(4) Prove that the circle described on a side of an isosceles triangle bisects the base.

(5) A system of right-angled triangles is described upon a given straight line as hypotenuse. Find the locus of the opposite angular points.

(6) Describe a square equal to the difference of two squares.

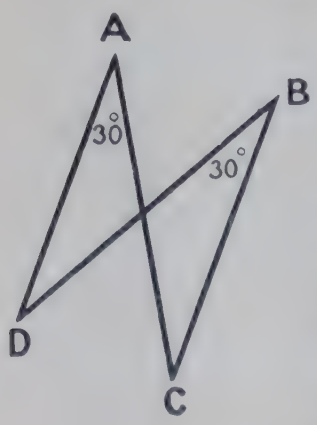
Concyclic Points.

Practical Exercises introducing Theorem 12.

(1) On the same side of a straight line AB construct two triangles ABD, ABC, having the angle at D equal to the angle at C. Prove by construction that the four points A, B, C, D lie on a circle.

(2) In a triangle ABC, BE, CF are altitudes. Show by construction that all the four points lie on a circle.

(3) Prove that all the triangles drawn on the same base and with equal vertical angles, have their vertices on an arc of a circle of which the given base is the chord.



(4) Draw a similar figure to the given diagram. Draw a circle round ABC; does it pass through D?

(5) Draw any circle and take four points A, B, C, D upon it. Find the sum of the angles BAC, BCD; and also the sum of the angles ABC, ADC.

(6) Draw four straight lines AC, CB, AD, DB, making angle C (65°) equal to angle D (see Fig. 4). Describe a circle round A, B, C, D. Does the circle pass through D?

Practical Exercises on Theorems 6 to 12.

(1) Draw a circle and circumscribe a parallelogram about it. Prove by measurement of sides and angles that the parallelogram is a rhombus.

(2) Prove that if two circles whose centres are A and B touch one another internally, and a straight line be drawn through their point of contact, cutting the circumference at C and D, the radii AC and BD are parallel.

(3) Draw two tangents to a circle, (*a*) at right angles, (*b*) at an angle of 120°.

(4) AB and AC are two tangents to a circle whose centre is D. Prove that AD bisects the chord of contact BC at right angles.

(5) Through a given point, within or without a circle, draw a chord equal to a given straight line.

(6) AB is any chord of a circle, AC the diameter through A, and AD the perpendicular on the tangent at B. Prove that AB bisects the angle DAC.

(7) Two chords intersect at A and B, and through A any straight line CAD is drawn terminated by the circumference. Show that CD subtends a constant angle at B.

(8) Show that the four circles drawn with the sides of a rhombus for diameters have one point in common.

(9) Through one of the points of intersection of two circles draw a chord of one circle which shall be bisected by the other.

(10) Draw a circle on one of the equal sides of an isosceles triangle as diameter. Prove by measurement that the circle passes through the middle point of the base.

(11) Find the angle in a segment of a circle, the chord of the segment being 5 cm. and the height 3 cm.

(12) Two circles intersect at P, Q. Through P diameters PA, PB of the two circles are drawn. Show that AQ, QB are in the same straight line.

Cyclic Quadrilaterals.

Practical Exercises introducing Theorem 13.

(1) Construct a quadrilateral by joining any four points in the circumference : find the sum of each pair of opposite angles.

(2) Repeat Exercise 1 on a larger scale. What do you conclude from your observations ?

(3) The side AB of a quadrilateral ABCD inscribed in a circle is produced to E. Prove that the exterior angle CBE equals the interior opposite angle ADC. Give the result in words.

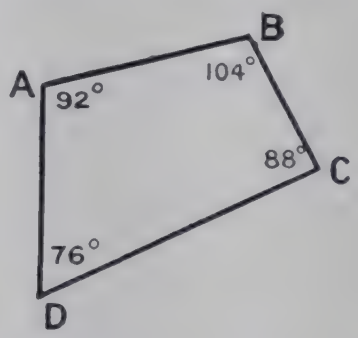
(4) In the circle ABC inscribe a parallelogram ABCD. Measure the sides and angles, and prove that the parallelogram ABCD is a rectangle.

(5) A quadrilateral ABCD is inscribed in a circle having the angle at A equal to 60°. E is the centre of the circle. Prove that the angles EBD, EDB equal the angles CBD, CDB.

(6) The sum of two opposite angles of a quadrilateral is equal to two right angles. Prove that the perpendiculars drawn to its four sides at their middle points meet in a point.

Practical Exercises introducing Theorem 14.

(1) Construct a quadrilateral ABCD, having its opposite angles supplementary, and prove that the quadrilateral is cyclic.



(2) Repeat Exercise 1, taking another quadrilateral with its opposite angles supplementary.

(3) Prove that if one side of a quadrilateral is produced, and the exterior angle so formed is equal to the interior opposite angle, then the quadrilateral is cyclic.

(4) If the exterior angles of a quadrilateral are bisected by four straight lines, prove that the quadrilateral formed by these lines is cyclic.

(5) BE, CF are two altitudes of a triangle ABC and intersect at G. Prove that AEGF is a cyclic quadrilateral.

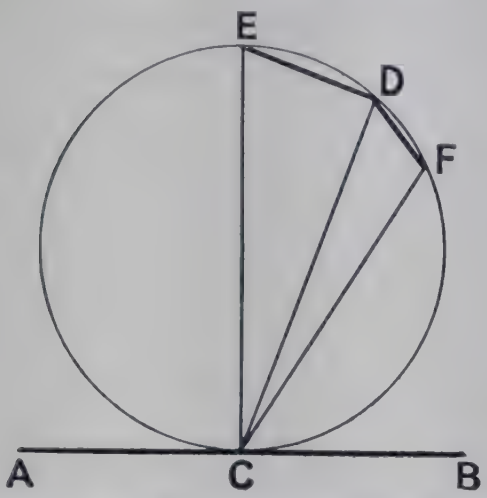
(6) Draw a quadrilateral ABCD, making the angles ABC, ADC equal to 180°. Describe a circle about ABC, and see whether the circle passes through D. What is your conclusion?

Angles in Alternate Segment.

Practical Exercises introducing Theorem 15.

(1) Show by experiment that if a straight line touch a circle, and from the point of contact a chord be drawn, the angles which this chord makes with the tangent are equal to the angles in the alternate segments—

i.e. ∠BCD = ∠ in alternate segment CED
∠ACD = ∠ in alternate segment CFD.



(2) Draw a segment of a circle which contains an angle of 50°, taking AB, 2·4 in. long, as chord. (See Problem 6.)

(3) Draw a segment of a circle containing an angle of 75°.

(4) At a given point A on a circle draw the tangent without finding the centre of the circle.

(5) In a circle of 1·5 in. radius inscribe a triangle having its sides parallel to three given straight lines.

(6) On a base 2·4 in. construct an isosceles triangle having a vertical angle of 80°.

Properties of Rectangles.

Practical Exercises introducing Theorem 16.

(1) If two chords of a circle intersect at a point inside the circle, the rectangle contained by the segments of the one is equal to that contained by the segments of the other. Prove the statement by measurement, etc.

(2) If two chords of a circle subsect at a point outside the circle, the rectangle contained by the segments of the one is equal to that contained by the segments of the other. Test the accuracy of this statement.

(3) Two circles intersect at A and B. Prove that AB produced bisects their common tangent.

(4) Two circles intersect at A and B, and through any point in AB, their common chord, two chords are drawn, one in each circle. Prove that their four extremities are concyclic.

(5) ABC is a triangle right-angled at C, and from C a perpendicular CD is drawn to the hypotenuse. Show that the square on CD is equal to the rectangle AD . DB.

(6) If two circles cut one another, the tangents drawn to the two circles from any point in the common chord produced are equal.

Miscellaneous Exercises.

(1) Describe a circle to touch three given straight lines.

(2) Describe a circle to pass through a given point and touch a given straight line at a given point.

(3) Given part of the circumference of a circle, find the centre and complete the circle.

(4) Trace the circumference of a coin, and find the centre of the circle traced out.

(5) Describe a circle to pass through a given point, and to touch a circle at a given point.

(6) Draw a circle to pass through a given point, and to touch a given straight line at a given point.

(7) Describe a circle of 1·5 in. radius to touch two circles of radii 1·3 in. and 1 in. respectively.

(8) Draw two transverse common tangents to two given non-intersecting circles.

(9) Draw two exterior common tangents to two circles.

(10) Draw segments of circles containing angles equal to (*a*) 60°, (*b*) 45°, (*c*) 84°, (*d*) 70° respectively on given straight lines.

(11) In a given circle inscribe a triangle equiangular to any given triangle.

(12) Draw a large scalene triangle. Inscribe a circle in the triangle.

(13) Describe an equilateral triangle about a given circle.

(14) Describe a circle to pass through two given points A, B 8 cm. apart and to have a radius of 6 cm.

(15) Put in a circle, end to end, six chords equal to the radius.

(16) Construct a triangle given the base 3 in., and one angle at the base 30°, radius of circumscribing circle 2 in. Find the magnitude of the vertical angle at A.

(17) Find the length of the minor and major arcs cut off from a circle of radius 6 cm. by a chord of 5 cm.

(18) Calculate the lengths of chords of a circle of radius 2 in., whose distances from the centre are (*a*) 1·5 in., (*b*) 1·2 in., (*c*) 1 in. Check results by measurement.

(19) In a square describe four equal circles.

(20) Describe a circle to pass through two given points.

(21) Draw a circle to touch three equal circles so as to enclose them all.

(22) Describe three equal circles, each touching the other two, and describe another circle round the other three.

(23) Draw tangents to a circle of radius 3 in. from a point 1·5 in. outside the circle. Calculate and measure the length of the tangents.

(24) Describe two circles of radius 2 in. and 1·4 in., having their centres 4 in. apart. Draw the two exterior common tangents.

(25) Draw a triangle on a base 4 in. long having a vertical angle of 70°.

(26) Construct a triangle on a base 4 in. having a vertical angle of 60°, and an area of 7·6 square inches.

(27) In a circle describe a triangle having its sides parallel to three given straight lines.

(28) On a base of 3 in. describe an isosceles triangle with a vertical angle of 56°.

(29) Inscribe a square in a circle of radius 3·9 cm.

(30) Circumscribe a square about a given circle of radius 5·4 cm.

(31) You are given the base, the altitude, and the radius of the circumscribing circle. Describe the triangle.

(32) Inscribe an equilateral triangle in a circle whose diameter is 2·4 in.

(33) Construct a triangle ABC given the angle at A, the side AB, and the length of the perpendicular from A to BC.

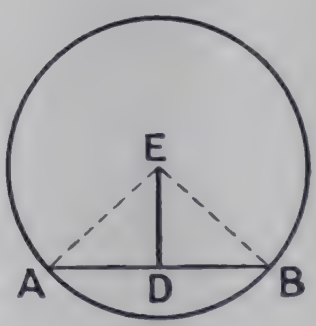
(34) Draw a circle of radius 1·5 in. Draw two tangents inclined at 60° to one another.

(35) Inscribe a circle in a given rhombus.

CHORD AND CENTRE.

Theorem 1.

A straight line drawn from the centre of a circle to bisect a chord which is not a diameter is at right angles to the chord; conversely, the perpendicular to a chord from the centre bisects the chord.



Let ED be a strt. line which joins E, the centre of ⊙ABC, to the middle point of the chord AB.

Then ED is ⊥ to AB.

CONSTRUCTION.—Join EA, EB.

PROOF.— In △s EAD, EDB,

EA = EB (radii),

ED common,

AD = DB (Hyp.),

∴ △s EAD, EBD are congruent (I. 14);

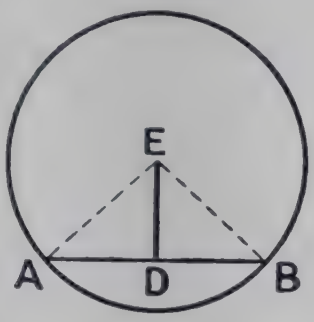
∴ adj. ∠ EDA = adj. ∠ EDB;

∴ ED is ⊥ to AB. Q.E.D.

Converse Proposition—

Let ED be a strt. line drawn from the centre E of ⊙ ABC, meeting the chord AB at rt. ∠s in D.

Then AD = DB.



CONSTRUCTION.—Join EA, EB.

PROOF.— In △s EAD, EBD,

∠ EDA = EDB (rt. ∠ s),

EA = EB (radii),

ED common,

∴ △s EAD, EBD are congruent (I. 17);

∴ AD = DB. Q.E.D.

COROLLARY.—A straight line which bisects the chord of a circle will, if produced, pass through the centre of the circle.

Exercises.

(1) Through a given point within a circle draw a chord which shall be bisected at that point.

(2) The perpendiculars bisecting any two parallel chords of a circle are in the same straight line.

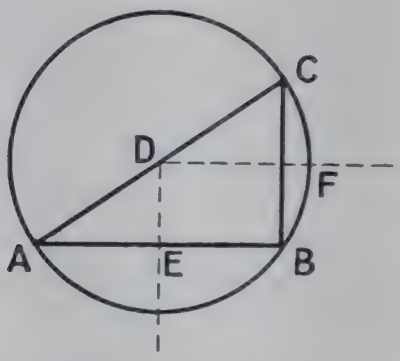
(3) Find the locus of the middle points of a system of parallel chords drawn in a circle.

(4) The perpendiculars bisecting the chords of a circle which do not pass through the centre are concurrent.

Circle through Three Points.

Theorem 2.

There is one circle, and one only, which passes through three given points not in a straight line.



Let A, B, C be three pts. not in the same strt. line.

Then only one ⊙, and one only, can be drawn through A, B, C.

Construction.—Draw DE, DF ⊥ bisectors of AB, BC respectively.

Proof.—Now the locus of a pt. equidistant from A and B is the ⊥ bisector DE.

Sim^{ly} the locus of a pt. equidistant from B and C is the ⊥ bisector DF.

But these bisectors intersect at D,

∴ D is equidistant from A, B, and C.

And there is no other pt. equidistant from A, B, and C.

∴ the ⊙ with centre D and radius DA passes through B and C; and this is the only one which can be drawn through A, B, and C. Q.E.D.

Corollary.—(1) Two circles cannot cut in more than two points.

(2) Circles which have three points in common coincide.

(3) The perpendicular bisectors of the chords of a circle meet in a point.

Exercises.

(1) If a circle passing through three given points has its centre in the straight line joining two of them, that line subtends a right angle at the third point.

(2) ABC, DBC are two triangles on the same base AB and on the same side of it. If AB is equal to BD, prove that AC and CD are unequal.

(3) A quadrilateral drawn within a circle has all its sides equal. Prove that all its angles are equal.

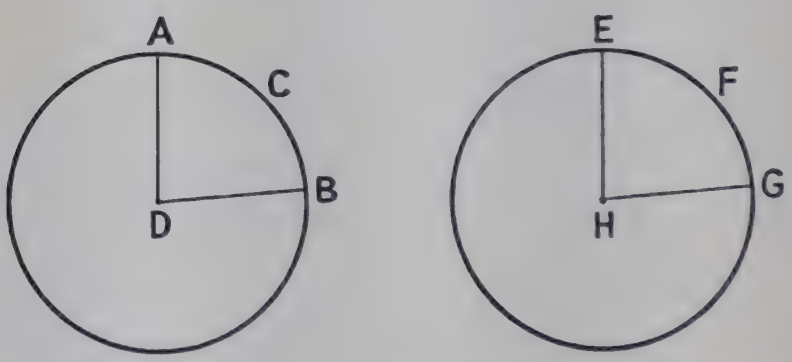
Congruence of Circles.

Theorem 3.

In equal circles (or in the same circle),

(*a*) *If two arcs subtend equal angles at the centres, they are equal.*

(*b*) *Conversely, if two arcs are equal, they subtend equal angles at the centres.*



(*a*) Let ABC, EFG be equal ⊙s, and the arcs ACB, EFG subtend equal ∠s ADB, EHG at the centres D, H.

Then arc ACB = arc EFG.

Proof.—Place ⊙ EFG on ⊙ ACB, so that H falls on D, and HG falls along DB,

∵ ∠ ADB = ∠ EHG (Hyp.),

∴ HE falls along DA.

Again, ∵ the ○ces of the ⊙s coincide,

∴ G coincides with B, and E with A;

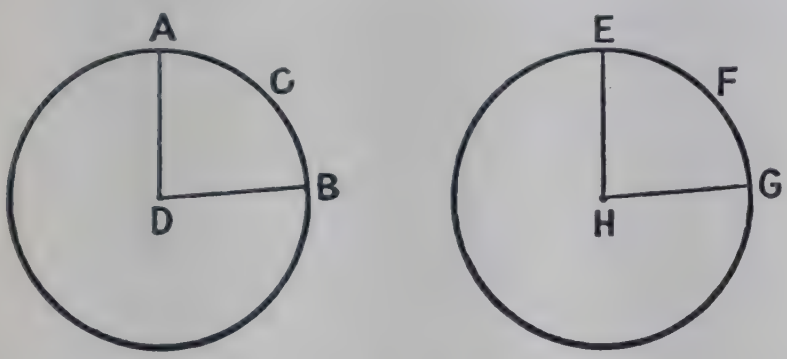
∴ arc EFG coincides with arc ACB;

∴ arc ACB = arc EFG. Q.E.D.

(*b*) *Converse Proposition*—

Let the arc ACB = arc EFG.

Then the ∠s ADB, EHG subtended by these arcs at the centres D, H are equal.



PROOF.—Place ⊙ EFG on ⊙ ACB, so that H falls on D, and HG falls along DB;

∵ the ⊙s are equal and the ◯ces coincide,

∴ G coincides with B,

∴ HG coincides with DB.

Then ∵ arc EFG = arc ACB,

E coincides with A,

∴ EH coincides with AD;

∴ ∠ ADB = ∠ EHG. Q.E.D.

Exercises.

(1) CAB are points on a circle whose centre is D. CA equals CB. Prove that C is the mid-point of the arc AB, and that DC bisects AB.

(2) C is a point on the circumference equidistant from the radii DA, AB. Prove that the arc AC equals the arc AB.

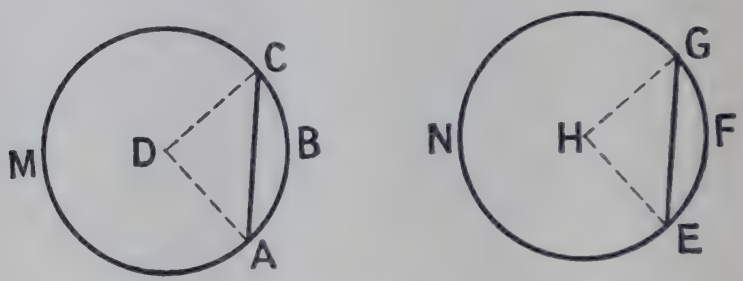
(3) If AB is an arc of a circle whose centre is D, and C any point on the remaining part of the circumference, prove that the angle ADB equals twice the angle ACB.

Theorem 4.

In equal circles (or in the same circle),

(*a*) *If two chords are equal, they cut off equal arcs.*

(*b*) *Conversely, if two arcs are equal, the chords of the arcs are equal.*



(*a*) Let ABC, EFG be equal ⊙s, whose centres are D and H, and chord AC = chord EG.

Then minor arc ABC = minor arc EFG,
and major arc AMC = major arc ENG.

CONSTRUCTION.—Join DA, DC; HE, HG.

PROOF.— In △s DAC, HEG,

AC = EG (Hyp.),

DA = HE, DC = HG } (radii of equal ⊙s),

∴ △s DAC, HEG are congruent (I. 14);

∴ ∠ ADC = ∠ EHG;

∴ arc ABC = arc EFG (III. 3).

As ◯ce of ⊙ ABC = ◯ce of ⊙ EFG,

∴ rem^{g} arc AMC = rem^{g} arc ENG. Q.E.D.

(*b*) *Converse*—

CONSTRUCTION.—Join DA, DC; HE, HG.

PROOF.— ∵ arc ABC = arc EFG (Hyp.),

∴ ∠ ADC = ∠ EHG (III. 3).

In △s ADC, EHG,

DA = HE, DC = DG } (radii of equal ⊙s),

∠ ADC = ∠ EHG,

∴ △s ADC, EFG are congruent (I. 7);

∴ chord AC = chord EG. Q.E.D.

COROLLARY.—The diameter is the greatest chord of a circle.

Exercises.

(1) A quadrilateral is inscribed in a circle, and AB equals CD. Show that AC equals BD.

(2) If two chords of a circle are parallel, they intercept equal arcs.

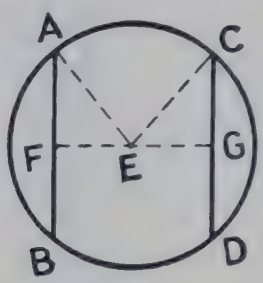
(3) If AB be any arc of a circle bisected in C, show that the chord AB is less than twice the chord AC.

(4) The straight lines which join the extremities of two equal arcs of a circle towards the same parts are parallel.

LENGTHS OF CHORDS.

Theorem 5.

Equal chords in a circle are equidistant from the centre; and, conversely, chords which are equidistant from the centre are equal.



Let ABC be a ⊙ with centre E.

Chord AB = chord CD,

and EF, EG ⊥ from E upon chords AB, CD

Then FE = GE.

CONSTRUCTION.— Join AE, CE.

PROOF.— ∵ EF is ⊥ to AB,

AF = FB (III. 1);

∴ AF = $\frac{1}{2}$AB.

In like manner, CG = $\frac{1}{2}$CD,

∴ AF = CG as AB = CD (Hyp.).

∵ ∠ AFE = rt. ∠ (Const.);

∴ $AE^2 = AF^2 + FE^2$.

Simly. $EC^2 = CG^2 + GE^2$.

But AE = EC
∴ $AE^2 = EC^2$ } Radii of same ⊙;

∴ $AF^2 + FE^2 = CG^2 + GE^2$.

But $AF^2 = CG^2$, for AF = CG (proved);

∴ $FE^2 = GE^2$;

∴ FE = GE. Q.E.D.

(*b*) *Converse Proposition*—

Let AB, CD be equidistant from the centre E,

i.e. FE = GE.

Then AB = CD.

PROOF.—Using the same construction as before,

$$AE^2 = AF^2 + FE^2,$$
$$CE^2 = CG^2 + GE^2,$$
$$\because AE^2 = CE^2 \text{ (radii of same } \odot);$$
$$\therefore AF^2 + FE^2 = CG^2 + GE^2.$$
$$\text{But } FE^2 = GE^2, \text{ for } FE = GE \text{ (Hyp.)};$$
$$\therefore AF^2 = CG^2;$$
$$\therefore AF = CG.$$
$$\text{But } AB = 2AF,$$
$$CD = 2CG;$$
$$\therefore CB = CD. \quad \text{Q.E.D.}$$

Exercises.

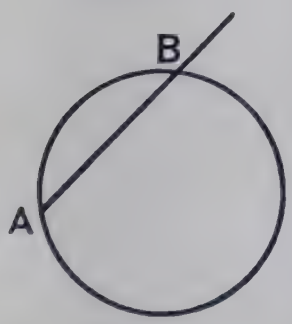
(1) If two equal chords intersect, prove that the segments of the one are equal respectively to the segments of the other.

(2) Show that the locus of the middle points of a set of equal chords of a circle is a concentric circle.

(3) If two chords of a circle make equal angles with the diameter through their points of intersection, they are equal.

(4) If two chords are at unequal distances from the centre, the nearer chord is longer than the more remote.

TANGENCY.



Looking at the figure, we see the secant AB cutting the circle at the points A and B.

Suppose the secant to be turned about the point A, so that the point B may gradually approach A, and ultimately coincide with it. When this limiting position has been reached, the

straight line AB has become the tangent to the circle at the point A.

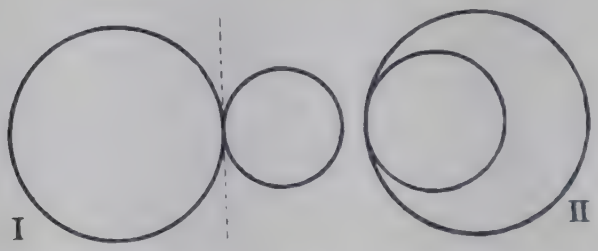
A **tangent** is a straight line whose two points of intersection with a circle are coincident.

The tangent is said *to touch* the circle, and the point at which it meets the circle is called *the point of contact.*

Two circles are said *to touch* one another at a point when they have a common tangent at that point, or

Two circles are said to touch one another when their two points of intersection coincide.

In Fig. I. the circles are said to touch *externally ;* in Fig. II. *internally.*

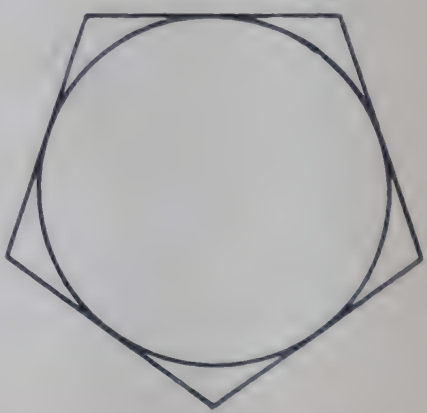


A rectilineal figure is said to be *inscribed* in a circle when the angular points of the figure are on the circumference of the circle.

A circle is said to be *circumscribed about* a rectilineal figure when the circumference of the circle passes through all the angular points of the figure.

A rectilineal figure is said to be *circumscribed about* a circle when each side of the figure touches the circle.

A circle is said to be *inscribed* in a rectilineal figure when the circumference of the circle is touched by each side of the figure.

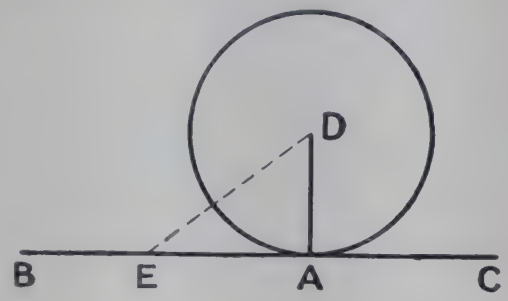


Theorem 6.

The tangent at any point of a circle is perpendicular to the radius through the point.

Let D be the centre of ⊙, and A any point on its ○ce.

Then tan. at A is ⊥ to radius DA.



CONSTRUCTION.—Through A draw BC ⊥ to DA.

Join D to any point E in BC.

PROOF.— ∵ DA is ⊥ to BC,

∴ DE > DA (I. 18);

∴ E is outside the ⊙.

Sim$^{ly.}$ every other pt. on BC, other than A, lies outside the ⊙,

∴ BC lies wholly outside the ⊙, and has only one pt. A in common with the ⊙;

∴ BC is tan. at A;

∴ tan. at A and radius AD are ⊥ to one another. Q.E.D.

COROLLARIES.—(1) A straight line drawn perpendicularly through the point of contact of a tangent to a circle passes through the centre of the circle.

(2) The straight line joining the centre of a circle to the point of contact of a tangent is perpendicular to the tangent.

Exercises.

(1) Prove that the point of intersection of the diagonals of a rhombus is equidistant from the four sides.

(2) The diameter of a circle bisects all chords which are parallel to the tangent at either extremity.

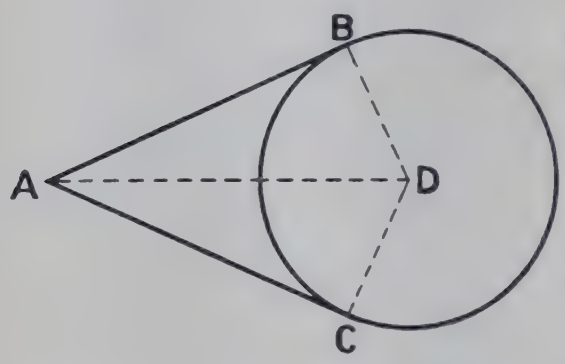
(3) All chords of a circle which touch an interior concentric circle are equal, and are bisected at the point of contact.

(4) A tangent of a circle meets two parallel tangents in A and B. Prove that AB subtends a right angle.

(5) If a parallelogram is circumscribed to a circle, it is a rhombus.

Theorem 7.

If two tangents are drawn to a circle from an external point,
(*a*) *they are equal;*
(*b*) *they subtend equal angles at the centre of the circle.*



Let AB, AC be tangents drawn to a ⊙, having its centre at D, from an external pt. A.

Then (*a*) AB = AC,
(*b*) ∠ ADB = ∠ ADC.

PROOF.— ∵ AB, AC are tangents at B, C,
∴ ∠s ABD, ACD are rt. ∠s (III. 6).

In the △s ABD, ACD,
DB = DC (equal radii),
AD common,
∠ ABD = ∠ ACD,
∴ △s ABD, ACD are congruent (I. 17);
∴ AB = AC,
∠ ADB = ∠ ADC. Q.E.D.

Exercises.

(1) A chord makes equal angles with the tangents at the extremities.

(2) The tangents at the extremities of a diameter are parallel to each other and to the chords bisected by that diameter.

(3) If a rectangle is circumscribed about a circle, it is a square.

(4) The angle between any two tangents to a circle is supplementary to the angle between the radii through the points of contact.

(5) Two parallel straight lines meet a third tangent at A, B. Prove that AB subtends a right angle at the centre.

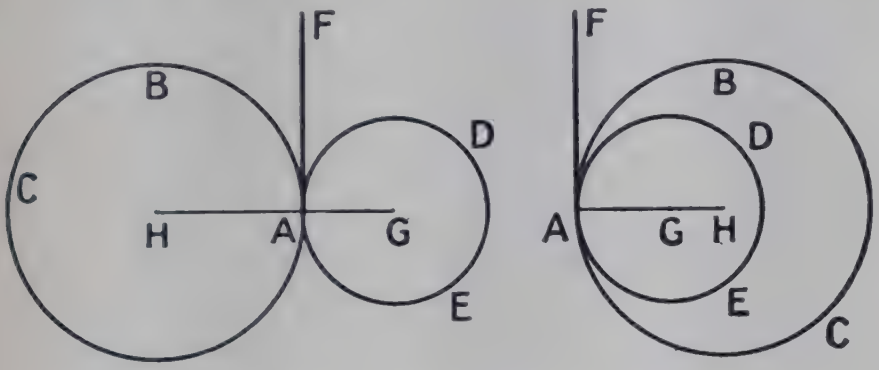
(6) The tangents to a circle from any point without it subtend equal angles at the extremities of the diameter through that point.

(7) In figure of Theorem 7 prove that the angles BAC, BCD are supplementary.

Contact of Circles.

Theorem 8.

If two circles touch, the point of contact lies on the straight line through the centres.



Let the two ⊙s, ABC, ADE touch either externally at A, or internally at A; let G, H be their centres, and AF the common tangent at A.

Join GA, HA.

Then HG passes through A.

Proof.—∵ ∠s GAF, HAF are rt. ∠s (III. 6),

∴ HAG is a st. line (I. 2). Q.E.D.

Exercises.

(1) If the distance between the centres of two circles is equal to the difference of their radii, then the circles must meet in a point.

(2) Determine the locus of the centres of all circles of a given radius, which touch a given circle.

(3) Two circles touch each other externally at A, and are touched by a straight line at B and C. Prove that BAC is a right angle.

Summary of Tangent Properties.

(1) The tangent is a straight line drawn at right angles to a diameter of a circle at one of its extremities.

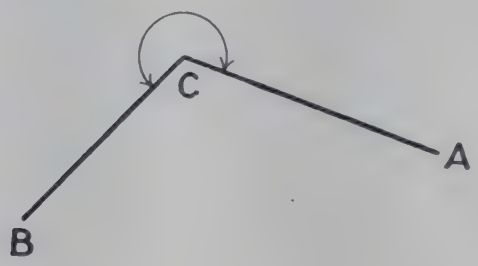
(2) It meets, but does not cut the circle.

(3) There can only be one tangent to a circle at a given point.

(4) The radius drawn to the point of contact is perpendicular to the tangent.

(5) Conversely, the line drawn perpendicularly to the tangent at the point of contact goes through the centre.

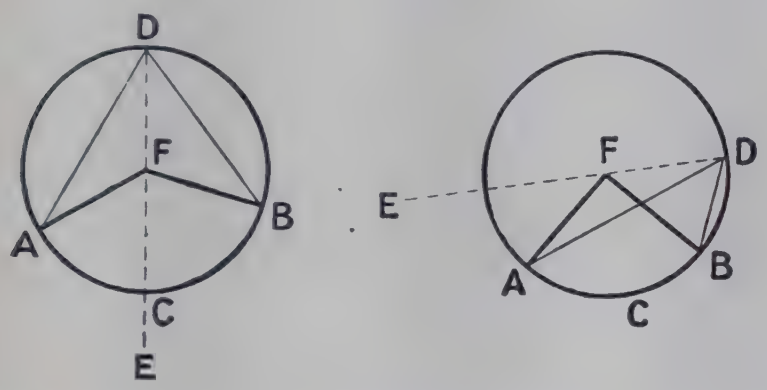
A **reflex** angle is an angle greater than two right angles, and less than four right angles.



Angles at Circumference.

Theorem 9.

The angle which an arc of a circle subtends at the centre is double that which it subtends at any point on the remaining part of the circumference.



Let arc ACB of ⊙ ACB subtend ∠ AFB at the centre F, and subtend ∠ ADB at D, any point on the remaining part of ⌒ce.

Then ∠ AFB = 2 ∠ ADB.

Construction.—Join DF, and produce to E.

Case I.—When the centre F is within the ∠ ADB.

In △ AFD,

AF = FD (equal radii),

∴ ∠ FAD = ∠ FDA (I. 8).

Ext. ∠ AFE of △ AFD = ∠ FAD + ∠ FDA (I. 6)

= 2 ∠ FDA.

Sim$^{ly.}$ ∠ EFB = 2 ∠ FDB,

∴ ∠ AFE + ∠ EFB = 2 (∠ FDA + FDB);

∴ ∠ AFB = 2 ∠ ADB.

Case II.—When the centre F is without the ∠ ADB.

As before, ext. ∠ AFE = 2 ∠ FDA,

and ext. ∠ EFB = 2 ∠ FDB,

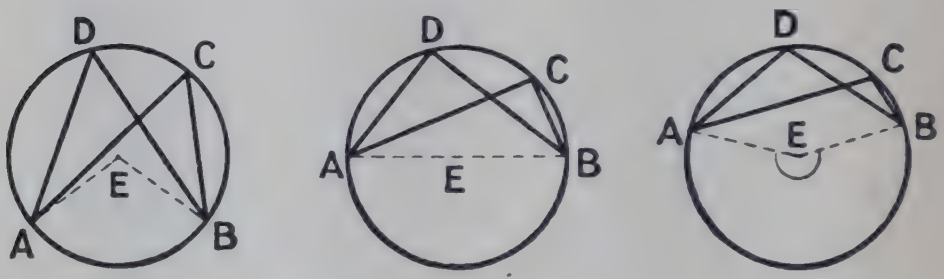
∴ ∠ EFB − ∠ AFE = 2 (∠ FDB − ∠ FDA);

∴ ∠ AFB = 2 ∠ ADB. Q.E.D.

Angles in a Segment.

Theorem 10.

Angles in the same segment of a circle are equal.



Let ∠ ACB, ADB be ∠s in the segment ABCD of ⊙ ABC; and let three cases be taken in which segment >, =, or < a semi-⊙.

Then ∠ ACB = ∠ ADB

Construction.— Join AE, EB.

Proof.—In each fig. ∠ AEB = 2 ∠ ADB (III. 9)

= 2 ∠ ACB,

∴ ∠ ACB = ∠ ADB. Q.E.D.

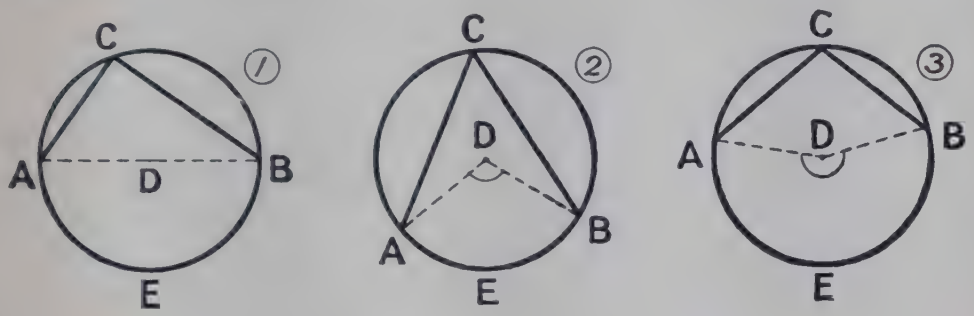
Exercises

(1) C is any point on the arc of a segment of which AB is the chord. Prove that the sum of the angles CAB, CBA is constant.

(2) AB and CD are two chords of a circle intersecting at F. Prove that the triangles AFC, DFB are equiangular.

Theorem 11.

The angle in a semicircle is a right angle; the angle in a segment greater than a semicircle is less than a right angle; the angle in a segment less than a semicircle is greater than a right angle.



Case I.—Let ABC be a semi-⊙.

Then ∠ ACB = rt. ∠.

PROOF.— ∵ ABC is a semi-⊙,

∴ strt. angle ADB = 2 rt. ∠s;

∴ ∠ ACB = rt. ∠ (III. 9).

Case II.—Let ABC be a segment > a semi-⊙.

Then ∠ ACB < rt. ∠.

PROOF.— ∵ ACB is a segment > a semi-⊙,

∴ arc AEB < a semi-⊙;

∴ ∠ ADB < 2 rt. ∠s.

But ∠ ACB = $\frac{1}{2}$ ∠ ADB;

∴ ∠ ACB < rt. ∠.

Case III.—Let ABC be a segment < a semi-⊙.

Then ∠ ACB > rt. ∠.

PROOF.— ∵ ACB is a segment < a semi-⊙,

∴ arc AEB > a semi-⊙;

∴ ∠ ADB > 2 rt. ∠s;

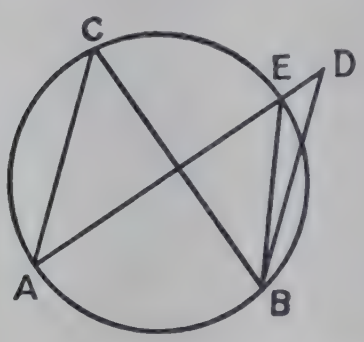
But ∠ ADB = 2 ∠s ACB,

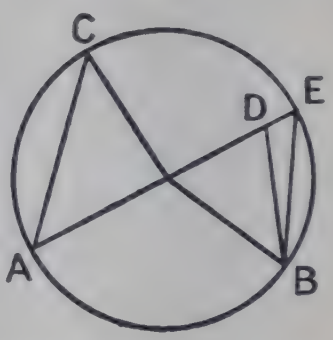
∴ ∠ ACB > rt. ∠. Q.E.D.

Concyclic Points.

Theorem 12.

If the line joining two points subtends equal angles at two other points on the same side of it, the four points lie on a circle.





Let the strt. line joining **AB** subtend equal angles at the pts. **C, D**, which are on the same side of **AB**.

Then the four pts., **A, B, C, D**, lie on a ⊙.

Construction.—Draw a ⊙ through **A, B**, and **C** (III. 2).

Proof.—If this ⊙ does not pass through **D** it will cut **AD** or **AD** produced in some pt. **E**.

Join **BE**.

Now ∠ **AEB** = ∠ **ACB** (III. 9).

But ∠ **ADB** = ∠ **ACB** (Hyp.),

∴ ∠ **AEB** = ∠ **ADB**, which is impossible (I. 6, Cor.) :

∴ ⊙ **ABC** must pass through **D**.

∴ **A, B, C, D** lie on a ⊙. Q.E.D.

Four or more points which lie on the same circle are said to be **concyclic.**

A rectilineal figure which is such that a circle can be drawn about it is said to be **cyclic.**

Exercises on Theorems 1–12.

(1) Describe a circle with a given centre cutting a given circle at the extremities of a diameter.

(2) If two circles cut each other, any two parallel straight lines drawn through the points of section to cut the circles are equal.

(3) Through a given point within a circle draw the shortest chord.

(4) Describe a circle which shall touch a given circle, have its centre in a given straight line, and pass through a given point in the given straight line.

(5) Draw parallel to a given straight line a straight line to touch a given circle.

(6) Draw a straight line to touch each of two given circles.

(7) Through either of the points of intersection of two given circles draw the greatest possible straight line terminated both ways by the two circumferences.

(8) ABD, ACE are two straight lines touching a circle at B and C, and if DE be joined, DE is equal to BD and CD together. Prove that DE touches the circle.

(9) Of all the triangles upon the same base and between the same parallels, the one that is isosceles has the greatest vertical angle.

(10) From the ends of a diameter BC of a circle, parallel chords BE, CF are drawn, meeting the circle again in E and F. Prove that EF passes through the centre of the circle.

(11) Prove that no parallelogram except a rectangle can be inscribed in a circle.

(12) The straight lines in a circle which join the extremities of two parallel chords are equal to each other.

(13) The chords of a circle which make equal angles with the straight line joining their point of intersection to the centre are equal.

(14) The circles described on the equal sides of an isosceles triangle as diameters will intersect at the middle point of the base.

(15) The greatest rectangle which can be inscribed in a circle is a square.

(16) Two circles intersect in A, B, and through the diameter BP, BQ are drawn. Prove that P, A, Q are collinear.

(17) If two chords of a circle meet at a right angle within or without a circle, the squares on their segments are together equal to the square on the diameter.

(18) If two circles cut one another, the tangents drawn to the two circles from any point in the common chord produced are equal.

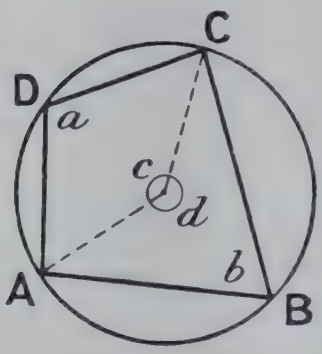
(19) Draw a trapezium in a circle, and show that the non-parallel lines are equal.

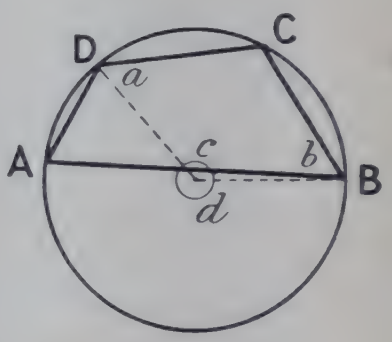
(20) A, B, C are three points in the circumference of a circle. Straight lines bisecting the angles CAB, ABC, BCA are drawn to meet the circumference again in D, E, F respectively. Prove that AD and EF are at right angles to one another.

Cyclic Quadrilaterals.

Theorem 13.

The opposite angles of any quadrilateral inscribed in a circle are supplementary.





Let **ABCD** be a quadrilateral inscribed in a ⊙.

Then ∠s **ABC, ADC** = 2 rt. ∠s,

∠s **DAB, DCB** = 2 rt. ∠s.

Construction.—Join **A** and **C** to the centre of the ⊙.

Proof.— $\angle a = \frac{1}{2} \angle d$ (III. 9),

$\angle b = \frac{1}{2} \angle c$ (III. 9),

$\therefore \angle a + \angle b = \frac{1}{2}(\angle c + \angle d)$.

But $\angle c + \angle d = 4$ rt. ∠s (I. Cor. 2),

$\therefore \angle a + \angle b = 2$ rt. ∠s;

$\therefore$ ∠s **ABC, ADC** = 2 rt. ∠s.

Sim^{ly}. ∠s **DAB, DCB** may be shown = 2 rt. ∠s. Q.E.D.

Exercises.

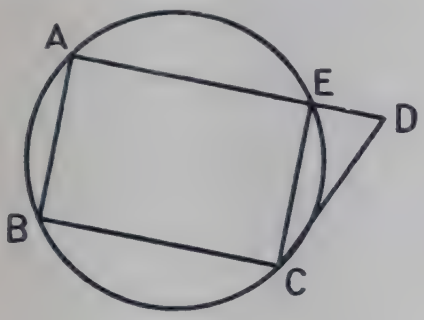
(1) If a circle can be described about a parallelogram, the parallelogram must be rectangular.

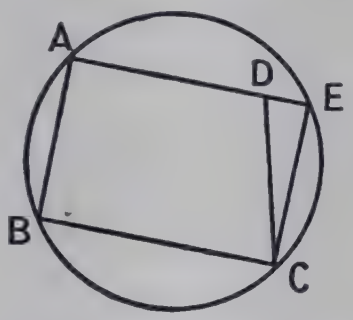
(2) If a quadrilateral be described in a circle, and a straight line be drawn making equal angles with one pair of opposite sides, it will make equal angles with the other pair.

Theorem 14.

(Converse of Theorem 13.)

If two opposite angles of a quadrilateral are supplementary, the quadrilateral is cyclic.





Let **ABCD** be a quadrilateral in which the ∠s **ABC, ADC,** are supplementary.

Then **ABCD** is cyclic.

CONSTRUCTION.—Draw a ⊙ passing through **ABC** (III. 2).

PROOF.—If this ⊙ does not pass through **D**, it will cut **AD**, or **AD** produced in some pt. **E**.

Join **EC**.

Now, ∠ **AEC** + ∠ **ABC** = 2 rt. ∠s (III. 13).

But ∠ **ADC** + ∠ **ABC** = 2 rt. ∠s (Hyp.),

∴ ∠ **AEC** + ∠ **ABC** = ∠ **ADC** + ∠ **ABC**,

∴ ∠ **ABC** = ∠ **ADC**, which is impossible (Cor. I. 6) ;

∴ **ABC** passes through **D** ;

∴ **ABCD** is cyclic. Q.E.D.

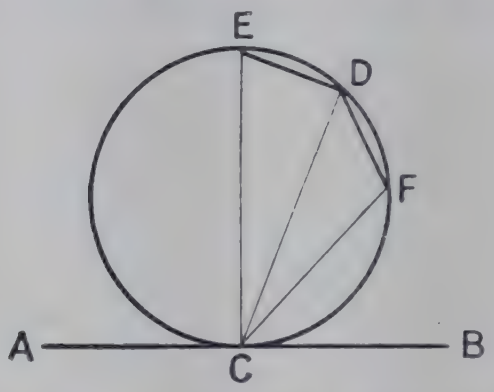
Exercise.

ABC, DCB are two congruent triangles on opposite sides of the base BC. Under what conditions are AB, CD concyclic?

Angles in Alternate Segments.

Theorem 15.

If a straight line touch a circle, and from the point of contact a chord be drawn, the angles which this chord makes with the tangent are equal to the angles in the alternate segments.



Let AB touch the ⊙ at C, and the chord CD be drawn from C, the point of contact.

Then ∠ BCD = ∠ in alt. segment CED,

∠ ACD = ∠ in alt. segment CFD.

Construction.—From C draw CE ⊥ to AB; take any pt. F in arc CFD, and join FC, ED, DF.

Proof.—∵ CE is ⊥ to tan. AB at C the pt. of contact,

∴ CE passes through the centre of the ⊙ and is a diameter (III. 6, Cor. 1).

∴ ∠ CDE = rt. ∠ (III. 11).

∴ in △ CED,

∠ CED + ∠ DCE = rt. ∠ (I. 5).

But ∠ BCD + ∠ DCE = rt. ∠ (Const.);

∴ ∠ BCD + ∠ DCE = ∠ CED + ∠ DCE;

∴ ∠ BCD = ∠ CED,

∵ CFDE is a quadrilateral inscribed in a ⊙,

∴ ∠ CED + ∠ CFD = 2 rt. ∠s (III. 13).

But ∠ BCD + ∠ ACD = 2 rt. ∠s (I. 1);

∴ ∠ BCD + ∠ ACD = ∠ CED + ∠ CFD;

But ∠ BCD = ∠ CED (proved),

∴ ∠ ACD = ∠ CFD. Q.E.D.

Exercises.

(1) B is a point in the circumference of a circle whose centre is C ; PA, a tangent at any point P, meets CB produced at A, and PD is drawn perpendicular to CB. Prove that the straight line PB bisects the angle APD.

(2) If two circles touch each other, any straight line drawn through the point of contact will cut off singular segments.

(3) Construct a triangle having given the base, the vertical angle, and the altitude.

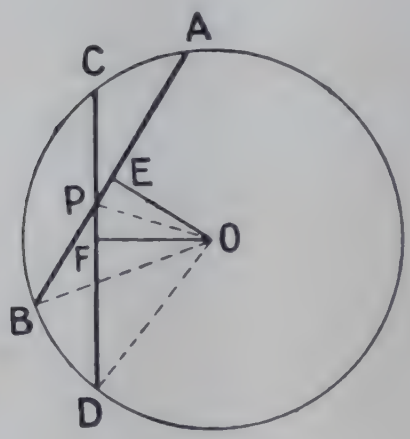
(4) AB is any chord, and AD is a tangent to a circle at A. DPQ is any straight line parallel to AB, meeting the circumference at P and Q. Show that the triangle PAD is equiangular to the triangle QAB.

(5) If from any point in the circumference of a circle a chord and a tangent be drawn, the perpendicular dropped on them from the middle point of the subtend arc are equal to one another.

Properties of Rectangles.

Theorem 16.

If two chords of a circle intersect at a point, either inside or outside the circle, the rectangle contained by the segments of the one is equal to that contained by the segments of the other.



Case I.—Let the chords AB, CD of the ⊙ ACB intersect in P within the ⊙.

Then rect. AP . BP = rect. CP . DP.

Construction.—Join OP, OB, OD.

From centre O draw OE, OF ⊥ AB, CD respectively.

Proof.—∵ AB and CD are bisected in E and F (III. 1), and divided unequally in P,

then $AP \cdot BP = EB^2 - EP^2$ (II. 6)

$= EB^2 + OE^2 - EP^2 - OE^2$

$= OB^2 - OP^2$ (II. 9)

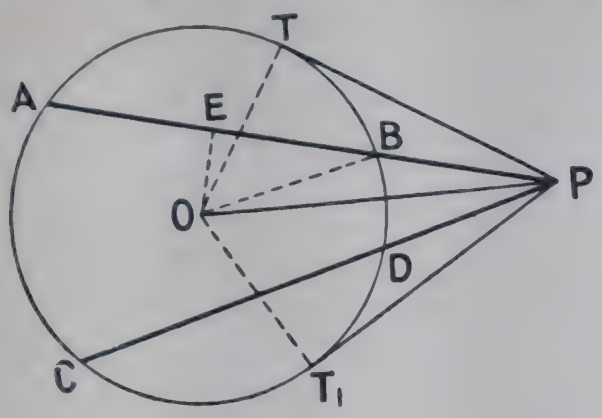
$= OD^2 - OP^2$

$= OF^2 + FD^2 - (OF^2 + FP^2)$

$= FD^2 - FP^2$

$= CP \cdot DP.$

Case II.—Let the chords AB, CD of the ⊙ ACB intersect in P without the ⊙, and let PT, PT_1 be the tangents from P.



CONSTRUCTION.—Draw OE ⊥ AB.

Join OT, OT_1.

PROOF.— ∠ OTP = ∠ OT_1P = rt. ∠ (III. 7),

and $AP \cdot BP = PE^2 - EB^2$ (II. 6)

$= PE^2 + OE^2 - (EB^2 + OE^2)$

$= OP^2 - OB^2$ (II. 9)

$= OP^2 - OT^2$

$= PT^2$.

Simly. $CP \cdot DP = PT_1^2 = PT^2$;

∴ rect. AP . BP = rect. CP . DP. Q.E.D.

COROLLARY.—If a tangent and a chord intersect, then the square on the tangent is equal to the rectangle contained by the segments of the chord.

Exercises.

Two straight lines, AB, CD, intersect at E, so that the rectangle AE, EB is equal to the rectangle CE, ED.

Prove that the four points A, B, C, D are concyclic.

Two circles intersect at A and B. Show that AB produced bisects their common tangent.

Loci.

Theorem 17.

The locus of the vertices of all triangles which have the same base and the sum of the squares of their sides equal to a given square is a circle having its centre at the middle point of the base.

Let ABC be a △ whose base is given, but whose vertex A moves in such a way that $AB^2 + AC^2$ is constant and equal to the sq. on PQ.

It is required to find the locus of A.

Construction.—Bisect BC at D; join AD; from A draw AE ⊥ BC or BC produced.

Now, of the two △s ADB, ADC one must be obtuse and the other acute.

Let △ ADB be acute.

Proof.—Then $AB^2 = AD^2 + BD^2 + 2BD \cdot DE$ (II. 11),

$AC^2 = AD^2 + CD^2 - 2CD \cdot DE$ (II. 12)

$= AD^2 + BD^2 - 2BD \cdot DE$;

$\therefore AB^2 + AC^2 = (AD^2 + BD^2 + 2BD \cdot DE) + (AD^2 + BD^2 - 2BD \cdot DE)$

$= 2(AD^2 + BD^2)$

$= PQ^2$ (Hyp.).

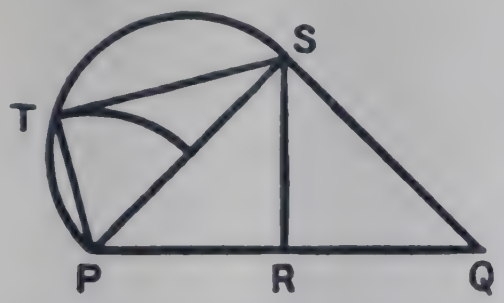
But PQ^2 and BD^2 are given;

$\therefore AD^2$ is also given;

$\therefore$ length AD is constant;

$\therefore$ locus of A is a ⊙, whose centre is D and whose radius = AD. Q.E.D.

N.B.—Bisect PQ at R; from R draw RS ⊥ PQ, making RS = PR = PQ; join PS, QS.



On PS as diameter describe a ⊙ cutting PTS at T; join PT, ST.

$\because$ PR = SR and $\angle$ PRS = rt. $\angle$,

$\therefore$ $\angle$ RSP = $\angle$ RPS = $\frac{1}{2}$ rt. $\angle$.

Sim$^{ly.}$ $\angle$ PSQ = $\angle$ RSP + $\angle$ RPS = rt. $\angle$.

Also $\angle$ RPS = $\angle$ RQS = $\frac{1}{2}$ rt. $\angle$,

$\therefore$ SP = SQ;

$\because$ $\angle$ PSQ = rt. $\angle$,

$\therefore$ $PQ^2 = PS^2 + QS^2$ (II. 9)

$= PS^2 + PS^2$

$= 2PS^2$.

But $\angle$ PTS being in a semi-⊙ = rt. $\angle$ (III. 11);

$\therefore$ $PQ^2 = 2PS^2 = 2(PT^2 + ST^2)$

$= 2(BD^2 + ST^2)$.

But $AB^2 + AC^2 = PQ^2$ (Hyp.);

$\therefore$ $AB^2 + AC^2 = 2(BD^2 + ST^2)$.

Also $AB^2 + AC^2 = 2(BD^2 + AD^2)$;

$\therefore$ $2(BD^2 + AD^2) = 2(BD^2 + ST^2)$;

$\therefore$ $AD^2 = ST^2$,

$\therefore$ AD = ST.

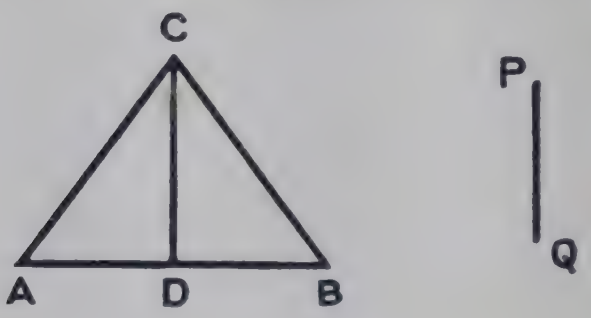
$\therefore$ the distance of the variable pt. A from the fixed pt. D, the middle pt. of BC, is constant, and equals ST where ST is given by $PQ^2 = 2(BD^2 + ST^2)$.

$\therefore$ the locus of A is a ⊙ whose centre is D, and whose radius equals ST, equals AD. Q.E.D.

Loci.

Theorem 18.

The locus of the vertices of all the triangles which have the same base and the difference of the squares of their sides equal to a given square is a straight line perpendicular to the base.



Let ACB be a $\triangle$ such that $AC^2 - BC^2 = PQ^2$, the pt. C being variable and PQ being given.

It is required to find the locus of C.

Construction.—From C draw CD $\perp$ the base AB.

Then $AC^2 = AD^2 + CD^2$ (II. 9),

$BC^2 = BD^2 + CD^2$ (II. 9);

$\therefore AB^2 - BC^2 = (AD^2 + CD^2) - (BD^2 + CD^2)$

$= AD^2 - BD^2$.

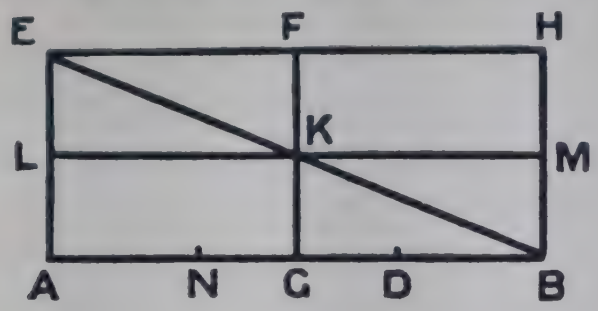
But $AB^2 - BC^2 = PQ^2$,

$\therefore AD^2 - BD^2 = PQ^2$, which is given.

$\therefore$ the pt. D is fixed, and $\therefore$ CD is a fixed straight line, since we have just shown that C always lies on it.

$\therefore$ the locus of C is a straight line perpendicular to AB through D, where D is given by $AD^2 - BD^2 = PQ^2$. Q.E.D.

N.B.—The pt. D can be found in the following way :—



CONSTRUCTION.—From A draw AE ⊥ AB, making AE = PQ.

On AE describe rect. AEFG, the pt. G being on AB.

From B draw BH ⊥ AB, cutting EF produced at H.

Join EB, cutting FG at K.

Through K draw LKM ‖ EH or AB, cutting AE at L and BH at M.

Now rect. FM = rect. LG.

To each of these add rect. EK ;

∴ rect. EM = sq. AEFG, *i.e.* = sq. on PQ ;

rect. EM is EH . HM, *i.e.* AB . HM.

From AB cut off AN = HM.

Bisect NB at D.

Then D is the required pt.

PROOF.—∵ $AD^2 - BD^2 = (AD + BD)(AD - BD)$

$= AB(AD - BD)$

$= AB(AD - ND)$

$= AB \cdot AN$

$= AB \cdot HM$

$= EH \cdot HM$

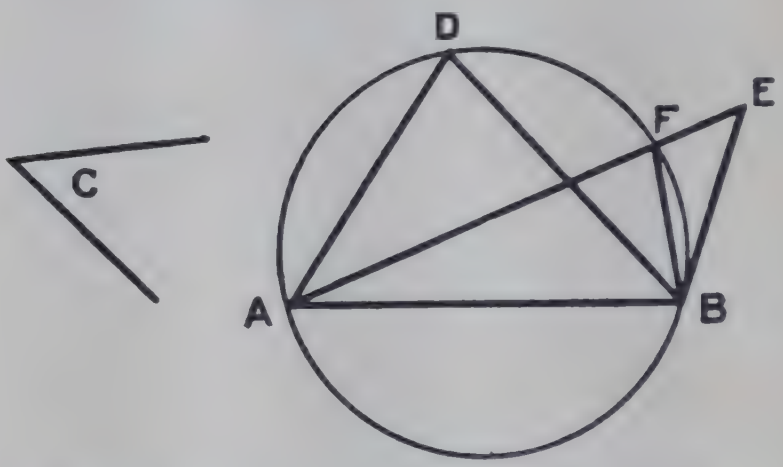
$= PQ^2$;

∴ D is the required pt.

Loci.

Theorem 19.

The locus of the vertices of all the triangles which have the same base, and their vertical angles equal to a given angle is the arc of a segment af a circle.



Let AB be the given base, ∠ C the given angle.

AEB is a △ standing on the fixed base AB, such that the ∠ AEB = ∠ C.

Then the locus of E is an arc of a segment containing an angle equal to the given angle.

Construction.—On AB describe the segment ADB such that ∠ ADB in the segment ADB = ∠ C.

Then E must lie on the arc ADB of this segment.

If not, let E be outside as in the figure, and let AE cut ADB at F. Join BF.

Proof.—Then ∠ AFB = ∠ ADB (III. 10)

= ∠ C.

But ∠ AEB = ∠ C;

∴ ∠ AFB = ∠ AEB, which is impossible.

∴ E cannot be outside as in the figure.

Sim$^{ly.}$ it can be shown that it cannot be inside.

∴ E must be on the arc ADB.

Sim$^{ly.}$ any other vertex like E will lie on the arc ADB.

∴ the locus of E is the arc ADB. Q.E.D.

Miscellaneous Exercises.

(1) Describe a circle through three given points. When is this impossible? Why?

(2) Show that the straight line which joins the mid-points of two parallel chords is at right angles to them.

(3) Through a given point within a circle draw a chord which is bisected at the given point.

(4) Through two given points describe a circle having its centre on a given straight line. When is this impossible?

(5) Prove that the "line of centres" bisects the common chord at right angles.

(6) Two circles intersect in A and B. Through A any straight line CD is drawn, terminated both ways by the circumference. Prove that the angle CBD is the same for every position of CD.

(7) Describe a circle with a given centre cutting a given circle at the extremities of a diameter.

(8) The lines bisecting the chords of a circle at right angles which do not pass through the centre are concurrent.

(9) A and B are two fixed points without a circle DEF. It is required to find a point D in the circumference, so that the sum of the squares described on AD and BD may be the least possible.

(10) Through a given point within a circle draw the shortest chord.

(11) Draw parallel to a given straight line a straight line to touch a given circle.

(12) Draw a straight line to touch each of two given circles.

(13) Given a point on the circumference of a circle, draw through it two chords at right angles to each other such that the distance of the one from the centre is twice the distance of the other.

(14) Through a given point draw a straight line so that the part intercepted by the circumference of a given circle shall be equal to a given straight line not greater than the diameter.

(15) Rectangles are the only parallelograms that can be inscribed in a circle.

(16) Through a given point within or without a circle draw a chord equal to a given straight line.

(17) The diameter of a circle bisects all chords which are parallel to the tangent at either extremity.

(18) A triangle is inscribed in a circle; show that the sum of the angles in the three segments exterior to the triangle is equal to four right angles.

(19) The straight lines bisecting any angle of a quadrilateral inscribed in a circle and the opposite exterior angle meet in the circumference of the circle.

(20) The bisectors of the angles formed by producing opposite pairs of sides of a cyclic quadrilateral are at right angles.

(21) Two circles touch each other externally at A, and are touched by a straight line at B and C. Prove that the angle BAC is a right angle.

(22) The sum of two opposite angles of a quadrilateral is equal to two right angles. Prove that the perpendiculars drawn to its four sides at their middle points meet in a point.

(23) AB and AC are a chord and a diameter meeting at a point A in the circumference. Prove that the radius drawn parallel to AB bisects the arc BC.

(24) Prove that a chord makes equal angles with the tangent at its extremities.

(25) Describe a circle to touch two given parallel straight lines and a given straight line which intersects them? How many solutions are there?

(26) Describe two circles touching each other externally at A and touched by a straight line at B and C. Prove that BAC is a right angle.

(27) D is any point on the arc of a segment of which AB is the chord. Prove that the sum of the angles DAB, DBA is constant.

(28) The side AB of a quadrilateral ABCD inscribed in a circle is produced to E. Prove that the exterior angle CBE equals the interior opposite angle ADC.

(29) BE, CF are two altitudes of a triangle ABC and intersect at G. Prove that AEGF is a cyclic quadrilateral.

(30) If two circles touch, any straight line through the point of contact cuts off similar segments.

(31) Two circles intersect in A and B, and through B the diameters BG, BH are drawn. Prove that BGAH are in the same straight line.

(32) Of two segments standing on the same chord, the greater segment contains the smaller angle.

(33) If two equal circles cut each other, and if through one of the points of intersection a straight line be drawn, terminated by

the circles, the straight lines joining its extremities with the other point of intersection are equal.

(34) The circles described on the equal sides of an isosceles triangle, as diameters intersect at the middle point of the base.

(35) Describe a square equal to the difference of two squares.

(36) If from the point of contact of a tangent to a circle a chord be drawn, the perpendiculars dropped on the tangent and chord from the middle point of either arc cut off by the chord are equal.

(37) ABC is a triangle right-angled at C; and from C a perpendicular CD is drawn to the hypotenuse. Prove that the square on CD is equal to the rectangle AD, DB.

(38) If two chords of a circle meet at a right angle within or without a circle, the squares on their segments are together equal to the square on the diameter.

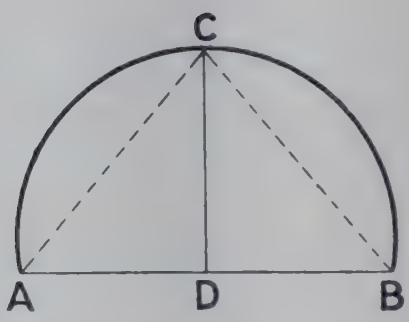
(39) If through any point in the common chord of two circles which intersect one another there be drawn any two other chords, one in each circle, their four extremities all lie in the circumference of a circle.

(40) A number of circles intersect each other, and are such that the tangents to them from a fixed point are equal. Prove that the straight lines joining the two points of intersection of each pair will pass through this point.

Constructions.

Problem 1.

To bisect a given arc.

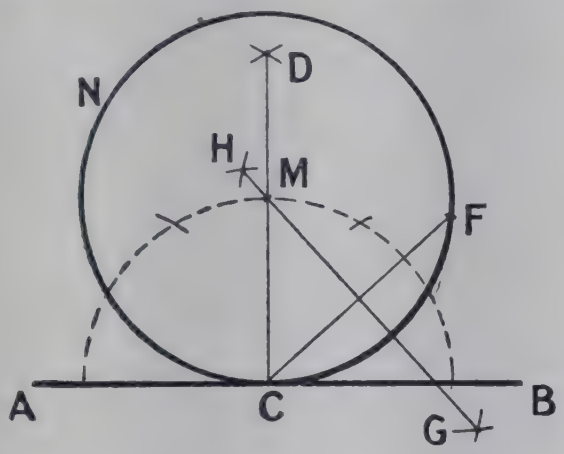


Let ABC be the given arc.
Join AB, and bisect it at D.
At D draw DC at rt. ∠ to AB, meeting the arc at C.
Then arc ABC is bisected at C.

Proof.— In △s ADC, BDC,
AD = BD,
CD common,
∠ ADC = ∠ BDC (Const.),
∴ AC = BC (I. 7);
∴ arc AC = arc BC (III. 3);
i.e. arc ABC is bisected at C. Q.E.F.

Problem 2.

To draw a circle to pass through a given point (F) *and to touch a given straight line* (AB) *at a given point* (C).



Let AB be the given straight line, and C the given point within it, and F the given point without it.

It is required to draw a circle passing through F and touching AB at C.

At C draw CD at rt. ∠ to AB. Join CF.

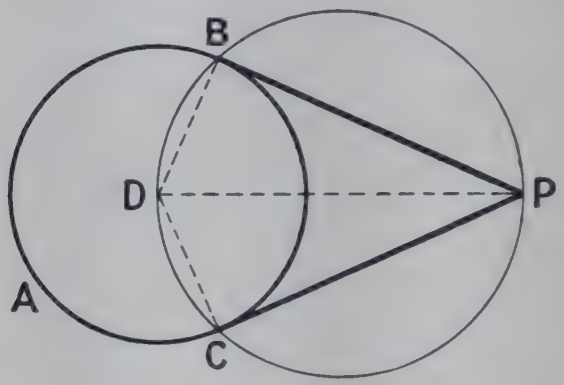
Bisect CF by the straight line GH.

Then the point of subsection of the two lines CD, GH is the centre of the ⊙ required. Describe the ⊙ CFN.

Give proof.

Problem 3.

To draw a pair of tangents to a circle from a given point (P) *outside.*

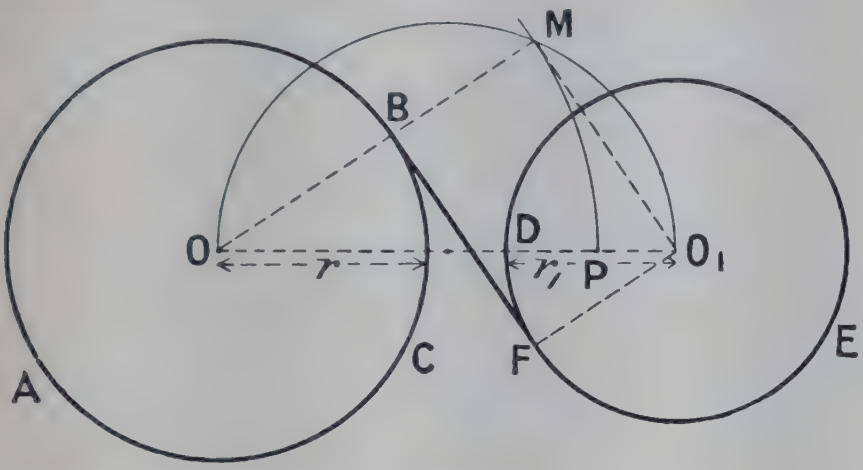


Let P be any point outside the ⊙ ABC.
Join D, the centre, and P.
Bisect DP, and describe a ⊙ on DP.
Join P to the points of intersection B and C.
Then PB and PC are the required tangents.

PROOF.— Join BD, CD.
∠ DBP = rt. ∠ (III. 11),
∠ DCP = rt. ∠,
∴ BP is a tangent (III. 6).
Sim$^{ly.}$ CP is a tangent. Q.E.F.

Problem 4.

To draw an interior common tangent to two given non-intersecting circles.



Let ABC, DEF be any two given ⊙s.

CONSTRUCTION.—Join their centres O and O_1.

Describe a semi-⊙ OMO_1 on the line of centres.

With centre O, and radii $(r + r_1)$, describe an arc PM cutting the semi-⊙ at M.

Join OM, MO_1.

Draw $O_1F \perp$ to MO_1.

Join FB.

Then FB is a common tangent to ⊙s ABC, DEF.

PROOF.—In quadrilateral BMO_1F,

$FO_1 \parallel$ to BM,

$\because$ ∠s BMO_1, MO_1F are rt. ∠s.

Again, $BM = r_1 = FO_1$

$\therefore$ BMO_1F is a $\square^m$ (I. 23).

And $\because$ ∠ BMO_1 is a rt. ∠,

$\therefore$ BMO_1F is a rectangle (I. 23, Cor. 2),

hence ∠ OBF is a rt. ∠,

$\therefore$ BF is a tangent to ⊙ ABC.

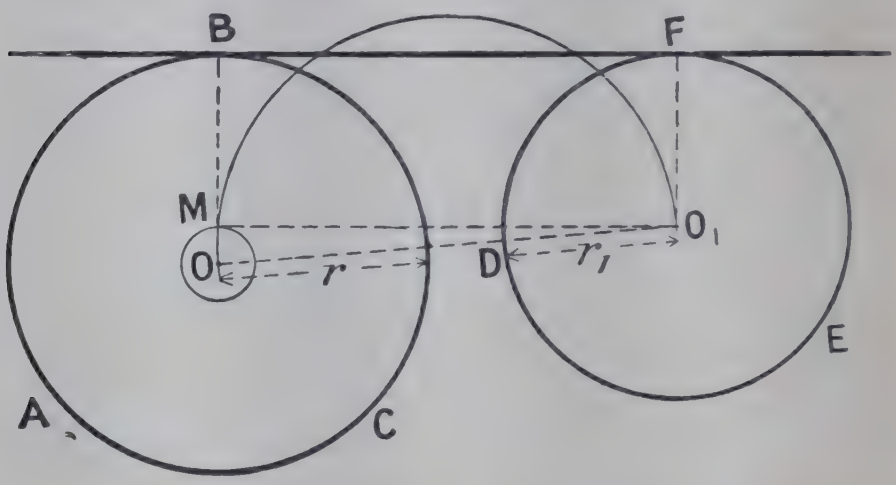
And ∠ BFO_1 is a rt. ∠,

$\therefore$ BF is a tangent to ⊙ DEF;

$\therefore$ BF is a common tangent to ⊙s ABC, DEF. Q.E.F.

Problem 5.

To draw exterior common tangent to two circles.

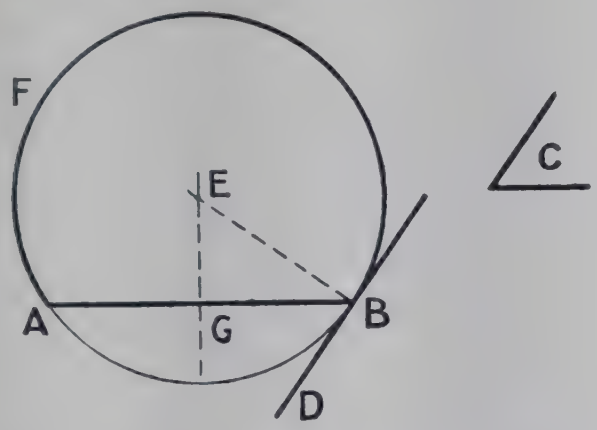


The construction is similar to the construction in the last problem, with the exception that the ⊙ described with centre O is of radius $r—r_1$.

Supply the proof.

Problem 6.

To draw a segment of a circle containing an angle equal to a given angle (C) *on a given straight line* AB.



Let AB be the given strt. line, and C the given ∠.

It is required to draw a segment of a ⊙ on AB containing an ∠ = ∠ C.

Construction.—At B make ∠ ABD = ∠ C. Erect a ⊥ at B, cutting the ⊥ bisector of AB at E.

This is the centre of a ⊙ passing through A and B.

With centre E and radius EB describe ⊙ ABF.

Proof.—The segment AFB of this ⊙ alternate to ∠ ABD is the segment required,

∵ EG bisects AB at rt. ∠s,

∴ EA = EB (I. 26);

∴ ⊙ passes through A and B,

and BD is a tan.

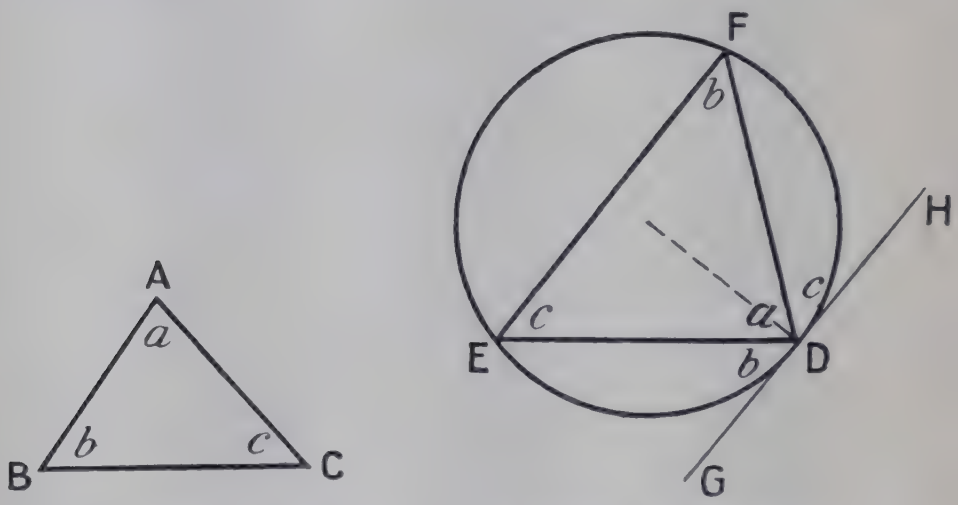
∵ EB is ⊥ to BD (Const.),

∴ ∠ ABD = ∠ in alt. segment AFB (III. 15),

i.e. ∠ C = ∠ in alt. segment AFB. Q.E.F.

Problem 7.

In a given circle inscribe a triangle equiangular to a given triangle ABC.



Let ABC be the given △, and DEF the given ⊙.

It is required to inscribe a triangle equiangular to the △ ABC in the ⊙ DEF.

Construction.—At D draw the tan. GDH.

Make ∠ EDG = ∠ DFE in alt. segment
= ∠ *b*.

Make ∠ HDF = ∠ DEF in alt. segment
= ∠ *c*.

Join EF.

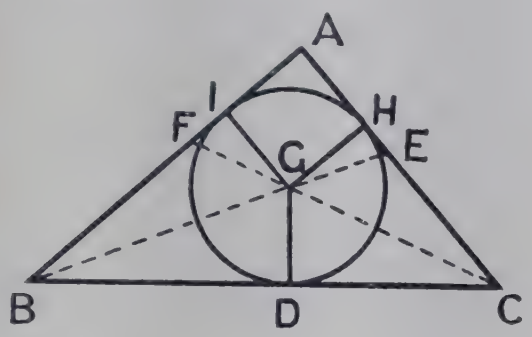
Then DEF is the required △.

Give proof.

CONSTRUCTIONS.

Problem 8.

To inscribe a circle in a given triangle.



Let ABC be the given △.

It is required to inscribe a ⊙ within the △ ABC.

CONSTRUCTION.—Bisect the ∠s ABC, ACB.

Draw DG ⊥ to BC to the pt. of intersection of the lines BE, CF.

With centre G and radius GD draw the ⊙ DHI.

Then DHI is the required ⊙.

PROOF.—∵ BG is the locus of a pt. equidistant from BC, BA (I. 26),

∴ GI = GD.

Sim^{ly}. GH = GD;

∴ ⊙ DHI passes through the pts. H, I,

and as ∠s at the pts. H, I, D are rt. ∠s,

∴ ⊙ touches AB, AC, BC. Q.E.F.

Circumference of a Circle.

Problem 9.

Draw several large circles; measure their circumference by thread and scale. Find also the diameters of the circles.

Tabulate your results as follows, and find how many times greater is the circumference than the diameter.

Experiment.	Circumference.	Diameter.	$\frac{\text{Circumference}}{\text{Diameter}}$
1	9·45 cm.	3 cm.	$\frac{9\cdot45 \text{ cm.}}{3 \text{ cm.}} = 3\cdot15$
2	8·9 cm.	2·8 cm.	$\frac{8\cdot9 \text{ cm.}}{2\cdot8 \text{ cm.}} = 3\cdot14$
3	12·5 in.	4 in.	$\frac{12\cdot5 \text{ in.}}{4 \text{ in.}} = 3\cdot13$

3)9·42

Average result = 3·14

Find the average result of your experiments. The correct result is 3·14 +, or $\frac{22}{7}$, which is generally expressed by the Greek letter π. Therefore the circumference of a circle is 3·14 +, or $\frac{22}{7}$ times greater than its diameter. In formula the statement is written $2\pi r$. The formula is thus obtained—

The circumference of a circle = 3·14 × diameter

= π × twice the radius

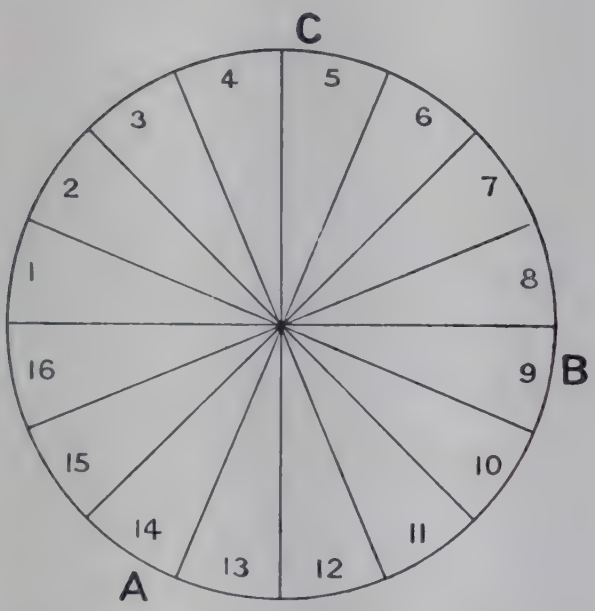
= $\pi \times 2r$

= $\pi\ 2r$

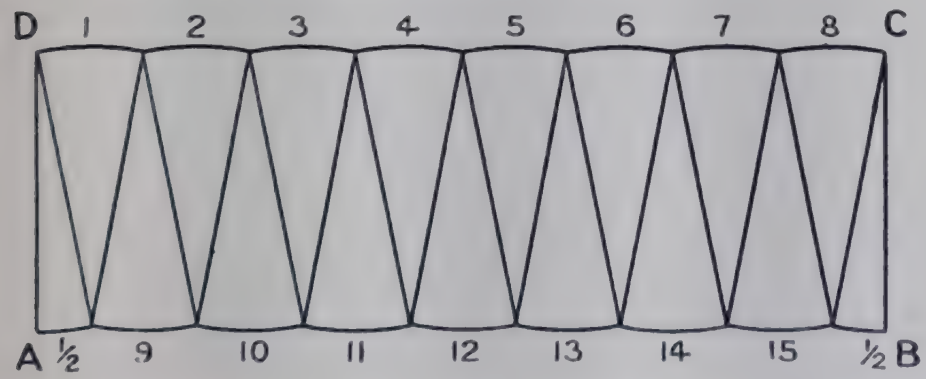
= $2\pi r$.

AREA OF A CIRCLE.

Problem 10.



On a piece of white paper describe a large circle ABC. Divide the circle into sixteen equal sectors. Cut out the sectors very carefully. Cut one of the sectors into two equal parts, and then arrange the sectors as below.



The sectors are so arranged as to form a rectangle. If the sectors were increased in number, the sides AB, CD would

more and more approach straight lines. We therefore assume AB, CD to be straight lines.

Then area of circle ABC = area of rectangle ABCD

$$= \text{length} \times \text{breadth}$$

$$= \frac{\text{circumference}}{2} \times \text{radius}$$

$$= \frac{2\pi r}{2} \times \text{radius}$$

$$= \pi r^2.$$

Exercises.

Find the area of the following:—

(*a*) half anna,
(*b*) quarter anna,
(*c*) any circle,
(*d*) base of a cylinder,
(*e*) base of a cone,
(*f*) any semicircle,
(*g*) rupee,
(*h*) base of your inkpot,
(*i*) circle of 4·6 cm. diameter,
(*j*) circle of 9·3 cm. diameter,
(*k*) circle of 2·25 cm. radius.

BOOK IV

This book is appended with a view to meeting the extra requirements of the Additional Mathematical Paper of the Calcutta Matriculation, London Matriculation, the Panjab Matriculation, and the Allahabad Matriculation Examinations.

H. A.

RATIO AND PROPORTION

In the first three books we have treated only of the absolute *equality* or *inequality* of magnitudes as tested by superposition. However, besides the relations of equality and inequality thus tested, there is another important kind of equality or inequality, viz. the relations which magnitudes may bear to one another, in respect of *quantity*, and not of the *magnitudes themselves*. This relation, in respect of quantity, is called **ratio**.

Ratio is the relation which one magnitude bears to another of the same kind with respect to quantity, the comparison being made by considering what multiple, part, or parts the first quantity is of the second. It should be noted that we cannot compare a length with an area, an area with a volume, or a length with a volume; but we can compare one length with another length, an area with another area, etc.

The ratio of a to b is expressed thus—

$$a : b, \text{ or } \frac{a}{b}, \text{ or } a \div b, \text{ or } a/b.$$

The quantities a and b are called the **terms** of the ratio; a is called the **antecedent** and b the **consequent** of the ratio.

The value of a ratio is measured by the fraction whose numerator is the antecedent, and denominator is the consequent. The ratio of $5 : 6$ is measured by the fraction $\frac{5}{6}$.

In the subsequent work ratios will be expressed in the fractional form as being the most convenient form to deal with.

Geometrical magnitudes of the same kind cannot always be expressed exactly in terms of a common measure or unit. The geometrical magnitudes are then said to be *incommensurable.*

Ex. The diagonal and the side of a square.
If a represents the side and b the diagonal, then—

$$b = \sqrt{2}a$$
$$\text{or } \frac{b}{a} = \sqrt{2}$$

Now $\sqrt{2}$ cannot be found exactly, thus the side and the diagonal are incommensurable. But if $a = 1''$, because $\sqrt{2} = 1{\cdot}414213$, then the diagonal $= 1{\cdot}414213''$, correct to $\frac{1}{100000}$ of an inch.

For practical purposes, therefore, all magnitudes may be considered commensurable, accuracy in the case of incommensurable magnitudes being obtained by the selection of an adequately small unit.

Proportion is the equality of ratio. When two ratios $a : b$ and $c : d$ are equal, then a, b, c, and d are said to be **proportionals.**

$$\text{If } \frac{a}{b} = \frac{c}{d},$$

a is to b as c is to d,

or thus: $a : b :: c : d.$

The terms a and d are called the **extremes**, and b and c are called the **means.**

If the four quantities a, b, c, and d be proportionals, the product of the extremes is equal to the product of the means—

$$\frac{a}{b} = \frac{c}{d}$$
$$\therefore ad = bc.$$

If a, b, c, and d represent the lengths of four straight lines, we obtain the following theorem :—

If four straight lines are in proportion, the rectangle contained by the extremes is equal to the rectangle contained by the means.

Quantities are said to be in **continued proportion** when the first is to the second, as the second is to the third, as the third is to the fourth, and so on. a, b, c, and d are in continued proportion when—

$$\frac{a}{b} = \frac{b}{c} = \frac{c}{d}$$

If three quantities a, b, and c are in continued proportion, b is said to be a **mean proportional** between a and c.

$$\text{If } \frac{a}{b} = \frac{b}{c} \qquad \therefore ac = b^2$$

That is, if three quantities are in continued proportion the product of the extremes is equal to the square of the mean.

If a, b, and c be in continued proportion, c is said to be a **third proportional** to a and b.

Some important relations easily deducible from the proportion $a : b :: c : d$ are given below for further reference.

$$\text{If } \frac{a}{b} = \frac{c}{d}$$

then (1) $\dfrac{b}{a} = \dfrac{d}{c}$ (Invertendo)

(2) $\dfrac{a}{c} = \dfrac{b}{d}$ (Alternando)

(3) $\dfrac{a+b}{b} = \dfrac{c+d}{d}$ (Componendo)

(4) $\dfrac{a-b}{b} = \dfrac{c-d}{d}$ (Dividendo)

(5) $\dfrac{a}{a-b} = \dfrac{c}{c-d}$ (Convertendo)

Practical Exercises introducing Theorem 1.

(1) Construct any large triangle ABC; draw a straight line DE parallel to the base BC, cutting the sides AB, AC in D and E. Prove by measurement and calculation that

$$\frac{AD}{DB} = \frac{AE}{EC}$$

(2) Describe a triangle ABC, and produce the sides BA, CA beyond the vertex A. Draw DE parallel to the base BC, cutting the sides BA, CA produced in D and E. Prove by measurement and calculation that DE cuts the two sides proportionally.

(3) Produce the sides AB, AC of any triangle below the base; draw a straight line DE parallel to the base BC, cutting the sides AB, AC produced in D and E. By measurement and calculation show that

$$\frac{AB}{AD} = \frac{AC}{AE}$$

What do you learn from the above three exercises?

(4) Describe any triangle ABC, and divide its sides AB, AC in the ratio 3 : 5 by the straight line DE.
Is DE parallel to BC?

(5) Produce the sides AB, CB of a triangle ABC beyond the vertex B, and divide the sides AB, CB produced proportionally by the straight line DE. Prove experimentally whether DE is parallel or not to AC.

(6) Draw a triangle ABC, produce the sides AB, AC below the base BC, and divide the sides AB, AC produced proportionally in D and E. Join DE. Determine experimentally whether DE is parallel or not to BC.
What conclusion do you draw from the last three exercises?

Practical Exercises introducing Theorem 2.

The sides of equiangular triangles, which are opposite to equal angles, are called **corresponding** sides.

(1) On the bases BC, EF, 2·4″, 1·8″ long respectively, describe two equiangular triangles ABC, DEF. By measurement and calculation prove that

$$\frac{BC}{EF} = \frac{CA}{FD} = \frac{AB}{DE}$$

(2) Describe a triangle ABC, given BC = 6 cm., CA = 5·4 cm., AB = 8 cm. On the base EF construct an equiangular triangle

DEF to triangle ABC. Prove that the corresponding sides of the two triangles are proportional.

From the two exercises what do you conclude?

(3) ABC is a triangle having AB = 1·5″, BC = 2″, CA = 2·5″. DEF is an equiangular triangle to the triangle ABC, having EF = 1″. Calculate DE, DF, and check result by measurement.

(4) Draw a triangle whose sides are in the ratio 3 : 4 : 5. (See Problem 24.)

(5) Construct two triangles ABC, DEF, having their sides taken in order proportional. Show that the triangles are equiangular and similar.

(6) ABC is a triangle right-angled at B ; prove that the altitude BD divides the triangle into two triangles which are similar to the triangle ABC. Set down the ratio properties you get from the similarity of the triangles BDA, BAC.

Practical Exercises introducing Theorem 3.

Equiangular triangles having their sides about each pair of equal angles, taken in the same order proportional, are said to be **similar.**

(1) Construct two triangles ABC, DEF, having one angle of the one equal to one angle of the other, and the sides about these equal angles proportional. By measurement prove that the triangles are equiangular, and hence similar.

(2) Draw two triangles ABC, DEF, having one angle (A) of the one equal to one angle (D) of the other, and the sides about these equal angles proportional, and let DEF be the smaller triangle. Along AB set off AG = DE, and along AC set off AH = DF. Join GH. Prove GH parallel to BC, and hence the two triangles. ABC, DEF are similar.

(3) Repeat the construction of the two triangles ABC, DEF as in Exercise 2. Produce CA to G, making AG = DE, and BA to H, making AH = DF. Prove GH is parallel to BC, hence the triangles AGH, *i.e.* DEF, and ABC are similar.

(4) In a right-angled triangle ABC, AD is drawn perpendicular to the base BC. Show that BD : DA = DA : AC, and that the triangles ABD, ADC are similar.

(5) If two straight lines AB, CD intersect in a point E, so that $\frac{AE}{ED} = \frac{CE}{EB}$, prove that ABCD are concyclic.

(6) The bases BC, EF of two similar triangles ABC, DEF are divided in the same ratio at H, G. Show that AH : DG = BC : EF.

Practical Exercises introducing Theorem 4.

Equiangular polygons having their sides about each pair of equal angles taken in the same order proportional, are said to be **similar.**

The following is a more practical definition. **Similar polygons** are those which can be built up of the same number of sides of equiangular triangles in the same way.

(1) (*a*) Construct a polygon on a base of 1·5″.
(*b*) Draw a similar polygon on a base of 1″. (See Problem 22.)

(2) Draw any polygon. On a base of 4 cm. draw a similar polygon, using the method employed in Problem 23.

(3) Describe two similar polygons ABCDE, *abcde*. Divide ABCDE into triangles by lines from X any point within the figure. In *abcde* at *a*, *b* make angles *bax*, *abx* equal to angles BAX, ABX respectively. Join *xc*, *xd*, *xe*. Prove that the polygons are divided into corresponding similar triangles.

(4) Construct two similar hexagons, and prove that by joining their vertices to a point within each, they can be divided into the same number of similar triangles.

(5) Draw two similar polygons ABCDE, *abcde* ; join DA, DB ; *da*, *db*. Prove that the polygons are divided into the same number of similar triangles.

(6) Construct two similar hexagons ABCDEF, *abcdef*, placing *abcdef*, the smaller, within the larger ABCDEF, with their corresponding sides parallel. Join each pair of corresponding points. Show that these lines pass through the same point, and that the two figures are divided into the same number of similar triangles.

Practical Exercises introducing Theorem 5.

(1) Construct two similar triangles ABC, EFG ; draw AD perpendicular to BC, and GH perpendicular to EF. Find the areas of the two similar triangles, and prove by measurement and calculation

$$\frac{\text{area of } \triangle \text{ ABC}}{\text{area of } \triangle \text{ EFG}} = \frac{BC^2}{EF^2}$$

(2) Draw two regular hexagons ABCDEF, *abcdef*, and prove that

$$\frac{\text{area of ABCDEF}}{\text{area of } abcdef} = \frac{AB^2}{ab^2}$$

Give conclusion drawn from the above two exercises.

(3) Describe equilateral triangles on the sides and diagonal of a square ; compare the ratio of their areas with the ratio of the lengths of their sides, and show that the sum of the areas of the two smaller ones equals the area of the other.

(4) The areas of two similar triangles are 100 sq. ins. and 64 sq. ins. ; the base of the greater is 7″. Find the base of the smaller.

(5) (*a*) What is the ratio of the areas of two similar triangles on bases of 4″ and 5″?

(*b*) Divide a triangle by a straight line parallel to the base into two equal parts. In what ratio is each side divided?

(6) Find the ratio of the bases of two similar triangles, one of which has double the area of the other.

Construct two such triangles, taking 2·5 cm. as the base of the smaller.

Practical Exercises introducing Theorem 6.

(1) Bisect the angle BAC of the triangle ABC by the straight line AD cutting BC in D. Prove that

$$\frac{DB}{DC} = \frac{AB}{AC}$$

(2) Draw any triangle ABC, and produce AB to F ; bisect the exterior angle CAF by the straight line AD drawn to meet BC produced at D. Prove that

$$\frac{DB}{DC} = \frac{AB}{AC}$$

(3) Describe any triangle, and show that the internal bisector of an angle of a triangle divides the opposite side internally in the ratio of the sides containing the angle ; and likewise the external bisector externally.

(4) In a triangle ABC, BC = 3″, CA = 3·5″, AB = 4″, and the internal bisector of the angle BAC cuts BC at D ; measure BD, DC ; check by calculation.

(5) In a triangle ABC, BC = 3″, CA = 4″, AB = 5″, and the external bisector of the angle CAF cuts the base produced at D. Measure BD, DC ; check by calculation.

(6) Prove that the bisectors of an angle of a triangle divides the base internally and externally in the same ratio.

Proportion.

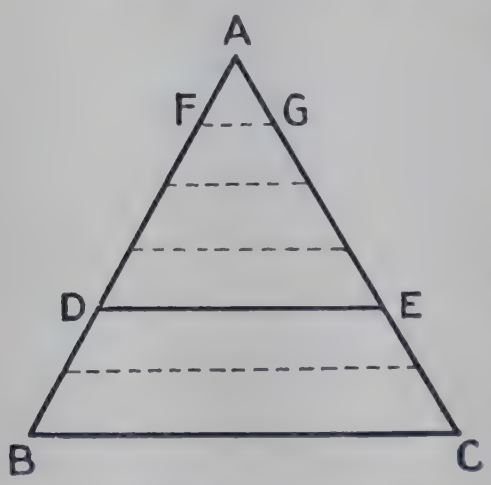
Theorem 1.

(*a*) *If a straight line is drawn parallel to one side of a triangle, the other two sides are divided proportionally;*

(*b*) *Conversely, if a straight line cuts two sides of a triangle proportionally, it is parallel to the third side.*

In $\triangle$ ABC let DE be a strt. line || BC, cutting AB, AC respectively, in D and E; then—

$$\frac{AD}{DB} = \frac{AE}{EC}$$



(*a*) Let AF be a common measure of AD, DB, and let $AD = xAF$ and $DB = yAF$, where x and y are integers.

Divide AD, DB into x and y equal parts respectively.

Through the points of division draw strt. lines || BC.

Proof.—These || lines divide AE into x, and EC into y equal parts.

$$\therefore AE = xAG \text{ and } EC = yAG;$$

$$\therefore \frac{AE}{EC} = \frac{x}{y}$$

$$\therefore \frac{AD}{DB} = \frac{AE}{EC}$$

(*b*) If DE be not || BC, suppose DE′ || BC; then DE and DE′ coincide.

Since DE′ || BC,

$$\therefore \frac{AD}{DB} = \frac{AE'}{E'C} \text{ (IV. 1).}$$

But $\frac{AD}{DB} = \frac{AE}{EC}$ (Hyp.);

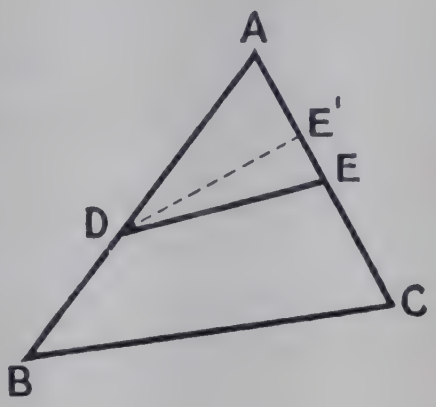
$$\therefore \frac{AE}{EC} = \frac{AE'}{E'C}$$

$$\therefore AE = AE';$$

∴ E and E′ coincide,

i.e. DE and DE′ coincide;

∴ DE || BC. Q.E.D.



Exercises.

(1) If any straight lines are cut by three parallel straight lines, they are cut proportionally.

(2) Triangles between the same parallels are to one another as their bases.

(3) Find a point within a triangle such that if straight lines be drawn from it to the three angular points, the triangle will be divided into three equal triangles.

(4) ABC is a triangle; it is required to draw from a given point P, in the side AB, or AB produced, a straight line to AC, or AC produced, so that it may be bisected by BC.

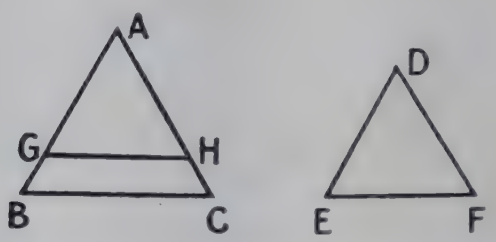
(5) AD, BC are the parallel sides of a trapezium. Prove that a line drawn parallel to these sides cuts the other sides proportionally.

EQUIANGULAR TRIANGLES.

Theorem 2.

(*a*) *If two triangles are equiangular, their corresponding sides are proportional;*

(*b*) *Conversely, if two triangles have their sides taken in order proportional, they are equiangular.*



(*a*) Let $\triangle ABC$, DEF have $\angle A = \angle D$, $\angle B = \angle E$, and $\angle C = \angle F$; then—

$$\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}$$

PROOF.—Apply $\triangle DEF$ to $\triangle ABC$ so that D may be on A and DE on AB; then FD falls on AC,

$$\because \angle D = \angle A.$$

Let E and F be at G and H respectively.

$$\because \angle AGH = \angle E = \angle B \text{ (Hyp.)};$$

$$\therefore GH \parallel BC \text{ (I. 21)};$$

$$\therefore \frac{AG}{AB} = \frac{AH}{AC}$$

$$\therefore \frac{AB}{AG} = \frac{AC}{AH}$$

$$\therefore \frac{AB}{DE} = \frac{AC}{DF}$$

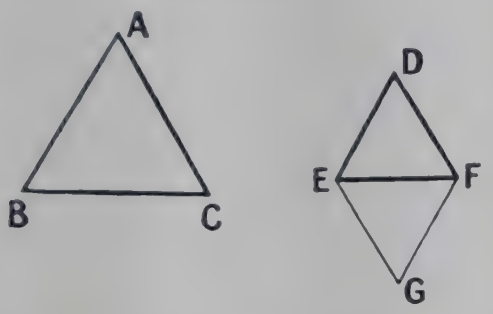
Sim$^{ly.}$ it may be proved that $\frac{AB}{DE} = \frac{BC}{EF}$.

$$\therefore \frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$$

(*b*) In $\triangle$s ABC, DEF, let—

$$\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF};$$

then $\triangle$s ABC, DEF are equiangular.



At E and F make $\angle$ GEF and $\angle$ GFE = $\angle$ B and $\angle$ C respectively.

Then $\triangle$ GEF, ABC are equiangular (Cor. I. 5).

$$\therefore \frac{AB}{GE} = \frac{BC}{EF} \text{ (IV. 2)}$$

$$= \frac{AB}{DE} \text{ (Hyp.)};$$

$$\therefore GE = DE.$$

$$\text{Sim}^{\text{ly.}}\ GF = DF,$$

and EF is common to both $\triangle$s DEF, GEF,

$\therefore$ $\triangle$s DEF, GEF are congruent and equiangular (I. 14).

And $\triangle$s ABC, GEF are equiangular,

$\therefore$ $\triangle$s ABC, DEF are equiangular. Q.E.D.

Exercises.

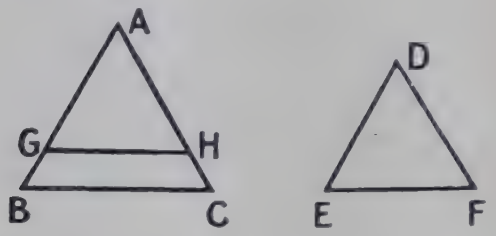
(1) In the sides AB, AC of a triangle ABC are taken two points DE, such that BD is equal to CE; DE, BC are produced to meet at F. Prove that AB : AC = EF : DF.

(2) If one of the parallel sides of a trapezium is double the other, show that the diagonals intersect one another at a point of trisection.

(3) A and B are two points on the circumference of a circle of which C is the centre. Draw tangents at A and B meeting at X, and from A draw AN perpendicular to CB. Show that BX : BC = BN : NA.

Theorem 3.

If two triangles have one angle of the one equal to one angle of the other, and the sides about these equal angles proportional, the triangles are similar.



Let △s ABC, DEF have ∠ A = ∠ D,

and $\frac{AB}{DE} = \frac{AC}{DF}$;

then the △s are similar.

Proof.—Apply △ DEF to △ ABC so that D may be on A, and DE on AB; then DF falls on AC.

$\because$ ∠ D = ∠ A.

Let E and F be at G and H.

Join GH.

$\because \frac{AB}{DE} = \frac{AC}{DF}$ (Hyp.),

$\therefore \frac{AB}{AG} = \frac{AC}{AH}$

$\therefore$ GH ∥ BC (IV. 1),

$\therefore$ ∠ B = ∠ AGH = ∠ E,

and $\therefore$ ∠ C = ∠ AHG = ∠ F;

$\therefore$ △s ABC, DEF are equiangular. Q.E.D.

Exercises.

(1) The bases BC, EF of two similar triangles ABC, DEF are divided in the same ratio at P, Q. Prove that AP : DQ = BC : EF.

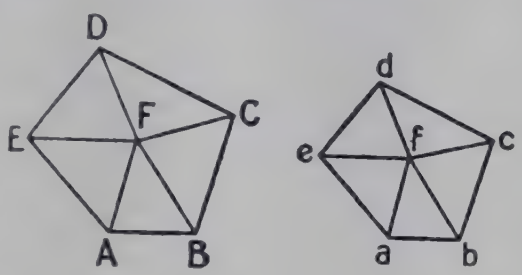
(2) In a triangle ABC, AD is drawn perpendicular to the base. Prove that the triangle ABC is right-angled if BD : DA = DA : DC.

SIMILAR POLYGONS.

Theorem 4.

If a polygon is divided into triangles by a line joining a given point to its vertices, any similar polygon can be divided into corresponding similar triangles.

Let ABCDE be a polygon divided into △s FAB, FBC, FCD . . . etc.; then any similar polygon *abcde* can be divided into corresponding similar △s.



PROOF.—Let AB, *ab*; BC, *bc*; CD, *cd*, etc., be pairs of corresponding sides.

At *a*, *b* make ∠s *fab*, *fba* = ∠s FAB, FBA respectively, and join *fc*, *fd*, *fe*.

Then since △s FAB, *fab* are equiangular,

$$\therefore \frac{FB}{fb} = \frac{AB}{ab} \text{ (IV. 2)}$$

$$= \frac{BC}{bc}$$

∵ the polygons are similar (Hyp.).

And ∵ ∠ ABC = ∠ *abc*

and ∠ ABF = ∠ *abf*

∴ ∠ FBC = ∠ *fbc*

Hence △s FBC and *fbc* are similar (IV. 3).

Sim$^{ly.}$ △s FCD, FDE, FEA are similar to △s *fcd*, *fde*, *fea* respectively. Q.E.D.

Exercises.

(1) If the given point be one of the vertices, B, prove the theorem.

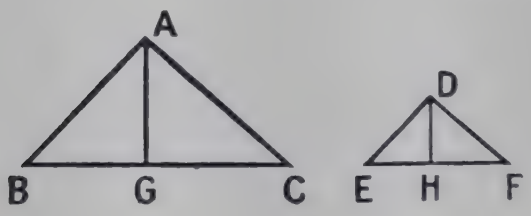
(2) Draw a triangle ABC congruent with a given triangle A′B′C′ so that the sides BC, CA, AB may pass through three given points P, Q, R.

Areas of Similar Figures.

Theorem 5.

The ratio of the areas of two similar triangles or of two similar polygons, is equal to the ratio of the squares on the corresponding sides.

(*a*) Let ABC, DEF be similar △s, having ∠ A = ∠ D, ∠ B = ∠ E, ∠ C = ∠ F; then—



$$\frac{\triangle ABC}{\triangle DEF} = \frac{BC^2}{EF^2}$$

Proof. — Draw AG, DH ⊥ BC, EF respectively.

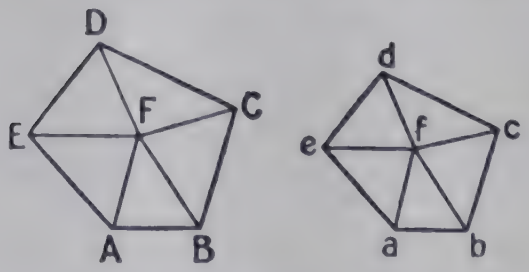
Then △s ABG, DEH are equiangular,

$$\text{and } \frac{AG}{DH} = \frac{AB}{DE} = \frac{BC}{EF} \text{ (IV. 2).}$$

$$\text{Now } \frac{\triangle ABC}{\triangle DEF} = \frac{\frac{1}{2}BC \cdot AG}{\frac{1}{2}EF \cdot DH} \text{ (I. 28, Cor.)}$$

$$= \frac{BC^2}{EF^2}$$

(*b*) Let ABCDE, *abcde* be similar polygons having AB, *ab*; BC, *bc*; CD, *cd*, etc., pairs of corresponding sides;

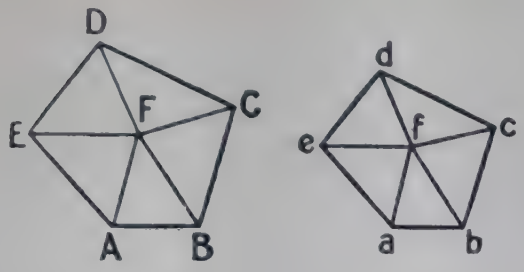


then fig. $\frac{ABCDE}{abcde} = \frac{AB^2}{ab^2}$

Proof.—

Within ABCDE take a pt. F.

Find a pt. *f* within *abcde* such that △s *fab*, *fbc*, *fcd*, etc., are respectively similar to △s FAB, FBC, FCD, etc. (IV. 4).



And since the polygons are similar—

$$\therefore \frac{AB}{ab} = \frac{BC}{bc} = \frac{CD}{cd} = \text{etc},$$

$$\text{and } \frac{\triangle FAB}{\triangle fab} = \frac{AB^2}{ab^2} = \frac{BC^2}{bc^2} \text{ (IV. 5}a\text{)}$$

$$= \frac{\triangle FBC}{\triangle fbc} = \text{etc.};$$

$$\therefore \frac{\text{sum of } \triangle\text{s in } ABCDE}{\text{sum of } \triangle\text{s in } abcde} = \frac{AB^2}{ab^2}$$

$$\therefore \frac{\text{fig. } ABCDE}{\text{fig. } abcde} = \frac{AB^2}{ab}. \quad \text{Q.E.D.}$$

Exercises.

(1) Prove that similar triangles are to one another in the same ratio as the squares on the radii of their circumscribing circles.

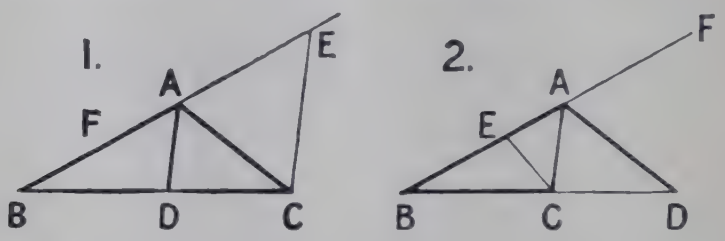
(2) Show that, if similar triangles are described on the three sides of a right-angled triangle, the area of the triangle described on the hypotenuse is equal to the sum of the other two triangles.

(3) Prove that the area of the regular hexagon inscribed in a circle is three-fourths of that of the regular hexagon described about the circle.

Bisector of Angle of Triangle.

Theorem 6.

(*a*) *The internal bisector of an angle of a triangle divides the opposite side internally in the ratio of the sides containing the angle ;*
(*b*) *And likewise the external bisector externally.*



Let ABC be a $\triangle$, and let AD bisect $\angle$ BAC internally as in Fig. 1, or externally (*i.e.* AD bisects $\angle$ CAF) cutting BC or BC produced in D ; then—

$$\frac{BD}{CD} = \frac{BA}{AC}.$$

Through C draw CE $\parallel$ DA to cut BA produced in E, or BA in E.

Since CE $\parallel$ AD,
$\angle$ FAD = $\angle$ AEC (I. 21),
$\angle$ DAC = $\angle$ ACE,
and $\angle$ FAD = $\angle$ DAC,
$\therefore$ $\angle$ AEC = $\angle$ ACE ;
$\therefore$ AE = AC.
Again, $\because$ AD $\parallel$ EC, a side of $\triangle$ BEC,

$$\therefore \frac{BD}{DC} = \frac{BA}{AE}$$

$$i.e. \frac{BD}{DC} = \frac{BA}{AC}. \quad \text{Q.E.D.}$$

Exercise.

If a straight line drawn from the vertex to the base of a triangle cuts the base internally or externally in the ratio of the sides, it bisects the vertical angle or its adjacent exterior angle. (Converse of Theorem 6.)

Miscellaneous Exercises.

(1) From a point E in the common base of two triangles ACB, ADB, straight lines are drawn parallel to AC, AD, meeting BC, BD at F, G. Prove that FG is parallel to CD.

(2) From any point in the base of a triangle straight lines are drawn parallel to the sides. Prove that the intersection of the diagonals of every parallelogram so formed lies in a certain straight line.

(3) The side BC of a triangle ABC is bisected at D, and the angles ADB, ADC are bisected by the straight lines DE, DF, meeting AB, AC at E, F respectively. Show that EF is parallel to BC.

(4) Divide a given straight line in a given ratio.

(5) Bisect a triangle by a straight line drawn parallel to one of its sides.

(6) Find a straight line such that the perpendiculars on it from three given points shall be in a given ratio.

(7) The diagonals of a quadrilateral, two of whose sides are parallel, cut one another in the same ratio.

(8) Find a point the perpendiculars from which, on the sides of a given triangle, shall be in a given ratio.

(9) ABCDE is a regular pentagon, and AD and BE intersect at F. Prove that AF : AE = AE : AD.

(10) Divide a given arc of a circle into two parts, so that the chords of these parts shall be to each other in a given ratio.

(11) Given two similar triangles, show how to construct a similar triangle.

(12) Show that the diagonals of a trapezium cut one another in the same ratio.

(13) If three straight lines drawn from a point cut two parallel straight lines in A, B, C and X, Y, Z respectively, prove that AB : BC :: XY : YZ.

(14) Divide a straight line into five equal parts.

(15) ABC is a triangle, PQ any straight line parallel to AC, cutting the sides BC, BA in P, Q respectively. Find the locus of the intersection of AP and CQ.

(16) If two triangles are on equal bases and between the same parallels, any straight line parallel to their bases will cut off equal areas from the two triangles.

(17) The chords AB, CD of a circle meet in X. Prove that whether X be within or without the circumference the triangles AXC, DXB are equiangular, and further prove that AX . XB = CX . XD.

(18) If two straight lines MN, PQ intersect in a point X, so that MX : XP = QX : XN, prove that M_1P_1NQ are concyclic.

(19) Prove that, if from the vertex of a triangle perpendiculars are drawn to the external bisectors of the angles at the base, the line joining the feet of these perpendiculars is parallel to the base.

(20) Prove that the hypotenuse of a right-angled triangle is to one side as the second side is to the perpendicular.

(21) ABCD is a quadrilateral, having the angles BAC, ACB right angles, and BM, DN perpendicular to AC. Prove that AN = CM.

(22) In unequal circles, similar segments stand upon chords proportional to their radii.

(23) If a triangle be divided into any two triangles by a straight line drawn from the vertex to the base, the radii of the circles circumscribed about the two triangles shall be in the ratios of the sides.

(24) A quadrilateral ABCD has the side AB equal to CD. F and F are the middle points of AC, BD. Show that EF is equally inclined to AB, CD.

(25) If in two circles two parallel radii are drawn, the straight line joining their extremities, produced both ways, will cut off similar segments of the circles, standing on chords which are to one another in the ratio of the radii.

(26) In unequal circles, similar segments stand upon chords proportional to the radii.

(27) ABCDE is a regular pentagon, and AD, BE intersect at O. Show that a side of the pentagon is a mean proportional between AO and AD.

(28) Divide a given arc of a circle into two parts, so that the chords of these parts shall be to each other in a given ratio.

(29) AB, CD are any chords of a given circle. If BC intersects AD at P, prove that the chords AB, CD are in the same ratio as their perpendicular distances from the point P.

(30) Form an equilateral triangle with its vertices on three given concurrent lines, and a side passing through a fixed point.

(31) Construct a triangle ABC congruent with a given triangle A'B'C so that the sides BC, CA, AB may pass through three given points.

(32) Describe a triangle, having given its angles and perimeter.

(33) The bisectors of the equal angles of two similar triangles are to one another as the bases of the triangles.

(34) If similar rectilineal figures are similarly described on the three sides of a right-angled triangle, the figure described on the hypotenuse is equal to the sum of the figures on the other two sides.

(35) Divide a triangle into two equal parts by a straight line at right angles to one of the sides.

(36) ABC, DEF are two triangles, in which AB:BC = DE:DF, and the angles BAC and EDF are supplementary. Prove that $\triangle ABC : \triangle DEF = AB^2 : DE^2$.

(37) AB is a straight line, and D is any point in it. Find a point P in AB produced such that PA : PB = DA : DB.

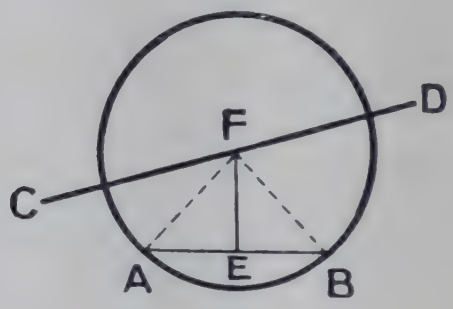
(38) Prove that the bisectors of an angle of a triangle divide the base internally and externally in the same ratio.

(39) In the triangle ABC the internal bisector of the angle BAC meets BC in D. Given the lengths of AB, AC, AD, show how to construct the triangle ABC.

(40) If the three sides of a right-angled triangle are in continued proportion, the perpendicular from the right angle on the hypotenuse divides it in extreme and mean ratio, that is medially.

Problem 1

To describe a circle passing through two given points and having its centre in a given straight line.



Let A and B be the given pts., and CD the given strt. line.

Join AB; bisect it in E and draw EF $\perp$ AB, meeting CD in F; join FA, FB.

Then F is the centre and FA the radius of the required $\odot$.

Proof.—$\because$ in $\triangle$s AEF, BEF,

AF = BF (I. 17),

$\therefore$ the $\odot$ described, with centre F and radius FA, passes through A and B, and has its centre in CD. Q.E.F.

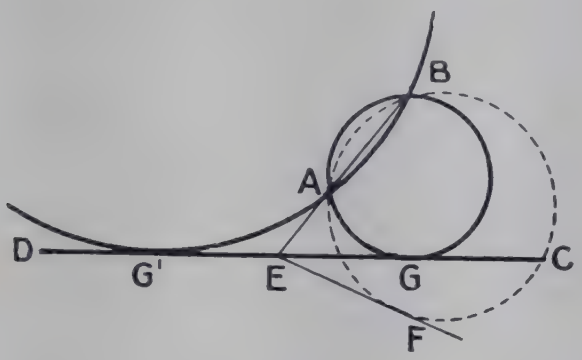
Exercises.

(1) Is the problem possible if CD is parallel to EF? Give reasons for answer.

(2) If CD coincides with EF, is the problem able to be demonstrated? Give reasons whether the problem is indeterminate or not.

Problem 2.

To draw a circle through two given points to touch a given straight line.



Let A and B be the two given pts., and CD the given strt. line. Join AB and produce the line AB to cut CD at E. Draw any ⊙ through A and B and let EF be a tan. to it. On EC, ED respectively make EG = EG′ each = EF. Then the ⊙s ABG, ABG′ are the ⊙s required.

Proof.—$EF^2 = EA \cdot AB$ = a constant for all ⊙s through A and B.

But $EF^2 = EG^2 = EG'^2$ (Const.)

$= EA \cdot AB.$

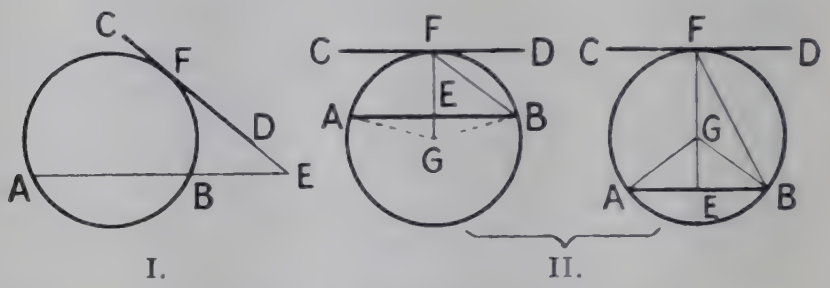
∴ since $EG^2 = EG'^2 = EA \cdot AB$, the ⊙s ABG, ABG′ passing through the pts. A and B touch the strt. line CD in G and G′ respectively. Q.E.F.

Exercise.

Try the above construction when AB is parallel to CD. Does the construction fail?

Problem 3.

To describe a circle passing through two given points and touching a given straight line.



I. II.

Case I.—Let A and B be the two given pts., and CD the given strt. line.

Produce CD to meet AB produced in E.

Make EF such that $EF^2 = AE \cdot EB$ (see Problem 14).

Describe a ⊙ passing through A, B, and F (III. 2).

Proof.—∴ the ⊙ ABF touches the strt. line CD in F;

$\because EF^2 = AE \cdot EB$.

Case II.—Let AB be ‖ CD.

Bisect AB in E; draw EF ⊥ CD; make ∠ FBG = ∠ BFG.

Then G is the centre and GB is the radius of the required ⊙.

Proof.—∴ the ⊙ ABF with centre G and radius GB passes through A and touches CD at F.

For GA = GB = GF,

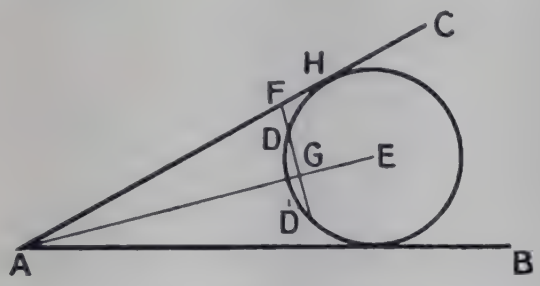
and ∠ GFD is a rt. ∠. Q.E.F.

Exercise.

If CD meets AB between A and B, is the problem possible?

Problem 4.

To describe a circle passing through a given point and touching two given straight lines.



Let D be the given pt., and AB, AC the given strt. lines.

Draw AE, bisecting the ∠ BAC.

From D draw DG ⊥ AE, and produce it to D′, making GD′ = DG.

Produce D′D to meet AC in F.

Describe a sq. = FD . FD′, and cut off FH = a side of the sq.

Describe a ⊙ through the pts. D′, D, H (III. 2).

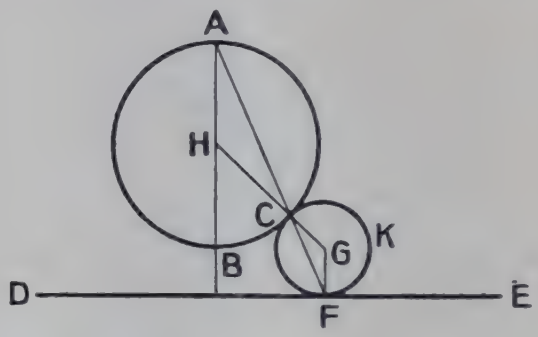
Proof.— Then since $FD \cdot FD' = FH^2$,

$\therefore$ ⊙ touches AC at H;

and ∵ its centre is in AE, the bisector of the ∠ BAC, it may be shown also to touch AB. Q.E.F.

Problem 5.

To describe a circle to touch a given circle, and also to touch a given straight line at a given point.



Let ABC be the given ⊙, DE the given strt. line, and F the given pt. in it.

From F draw FG ⊥ DE.

Then the centre of the required ⊙ must lie in FG.

Find H, the centre of the ⊙ ABC (III. 1).

Draw AHB, a diameter ⊥ DE; join F to one extremity A, cutting the ◯ce at C.

Join HC, and produce it to cut FG at G.

With radius GF describe the ⊙ FKC.

Prove that it touches the given ⊙ ABC at C, and the given strt. line DE at F.

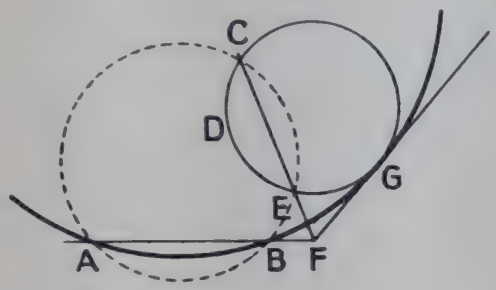
Exercises.

(1) Show that a second solution is obtained by joining FB, and producing it to meet the circumference.

(2) Describe a circle to touch a given straight line, and to touch a given circle at a given point.

Problem 6.

To describe a circle to pass through two given points and to touch a given circle.



Let A and B be the given pts., and CDE the given ⊙.

Through A and B describe any ⊙ cutting the given ⊙ CDE in C and E. Join AB, CE, and produce them to meet in F.

From F draw FG tan. to the ⊙ CDE, and let G be the pt. of contact.

Then the ⊙ described through ABG will touch the given ⊙.

PROOF.— Since in the ⊙ ABEC,

$$AF \cdot FB = FC \cdot FE;$$

and in the ⊙ DCE,

$$FG^2 = FC \cdot FE;$$

$$\therefore AF \cdot FB = FC \cdot FE = FG^2;$$

$\therefore$ FG touches the ⊙ ABG at G.

$\therefore$ the ⊙ ABG touches the given ⊙, and it passes through the given pts. A and B. Q.E.F.

Exercises.

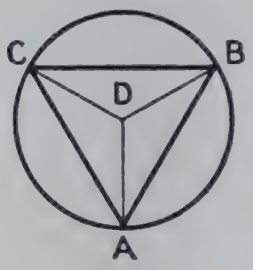
(1) How many solutions are there of this problem? Why?

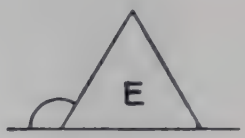
(2) Does the general construction fail when the straight line bisecting AB at right angles passes through the centre of the given circle?

(3) Draw a circle with its centre on a given straight line, to pass through a given point, and to touch a given circle.

Problem 7.

To inscribe an equilateral triangle in a given circle.





Let **ABC** be the given ⊙.

Find its centre **D**.

Draw any equil. △ **E**, and produce one of its sides.

At **D**, in the ⊙ **ABC**, make the ∠s **ADB, BDC, CDA** = ext. ∠ of equil. △ **E** = 120°.

Join **AB, BC, CA**, the pts. on the ○ce.

Then **ABC** is the required △.

PROOF.—Since ∠ **ADB** = ∠ **BDC** = ∠ **CDA**,

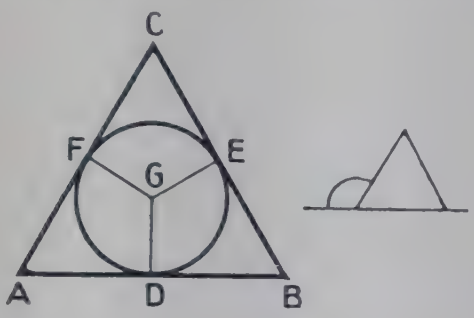
∴ arc **AB** = arc **BC** = arc **CA** (III. 3);

∴ chord **AB** = chord **BC** = chord **CA** (III. 4),

∴ **ABC** is an equil. △. Q.E.F.

Problem 8.

To circumscribe an equilateral triangle about a given circle.



Let **DEF** be the given ⊙.

Find **G**, its centre (III. 1).

At centre **G** make ∠s **DGE, EGF, FGD** each = 120° = ext. ∠ of an equil. △, and let their arms terminate on the ◯ce at **D, E, F**.

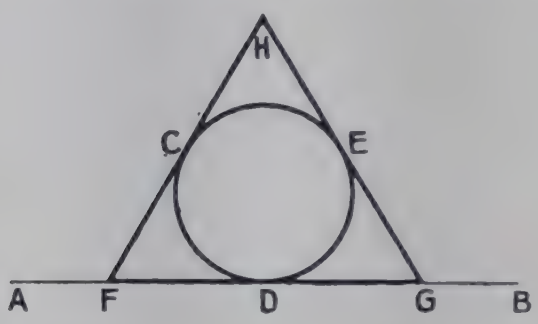
Through the pts. **D, E, F** draw the tans. **AB, BC, CA** to the given ⊙, cutting each other in the pts. **A, B, C** (III. 6).

Then **ABC** is the required equil. △.

Give proof.

Problem 9.

To circumscribe an equilateral triangle about a given circle.



Let **CDE** be the given ⊙.

Draw a tan. **AB** to the ⊙ **DEF** (III. 6).

With a set square having an ∠ 60°, draw the tan. **HF** to the ⊙, making an ∠ 60° with **AB**, or **AB** produced at **F**.

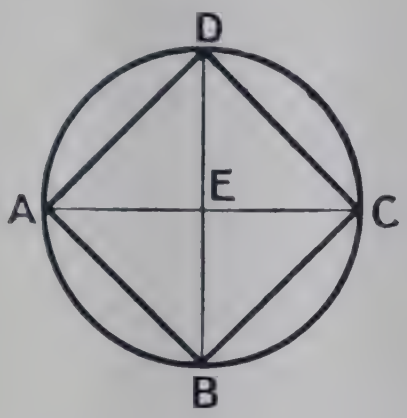
Sim^{ly}. draw the tan. **HG** to the ⊙, making an ∠ 60° with **AB**, or **AB** produced at **G**.

Then **FGH** is the required equilateral △.

Give proof.

Problem 10.

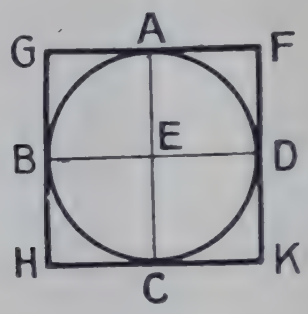
To inscribe a square in a given circle.



Let **ABCD** be the given ⊙.
Find **E**, the centre of the ⊙ (III. 1).
Draw any diameter **AC**, and another **BD** ⊥ to it.
Join **AB, BC, CD, DA.**
Then **ABCD** is the required sq.
Give proof.

Problem 11.

To circumscribe a square about a given circle.



Let ABCD be the given ⊙.

Find E the centre of the ⊙, and draw two diameters AC, BD ⊥ to one another.

Through A, B, C, D draw FG, GH, HK, KF ⊥ EA, EB, EC, ED.

Then FGHK is the required sq.

Give proof.

Exercises.

(1) Circumscribe a circle about a given square.

(2) Inscribe a circle in a given square.

(3) Circumscribe a rhombus about a given circle.

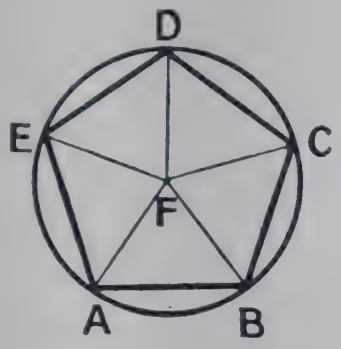
(4) Describe a circle about a given rectangle.

(5) Describe a square about a given rectangle.

(6) Describe (i) a square ; (ii) a circle in a given quadrant.

Problem 12.

To inscribe a regular pentagon in a circle.



Let ABCDE be the given ⊙.

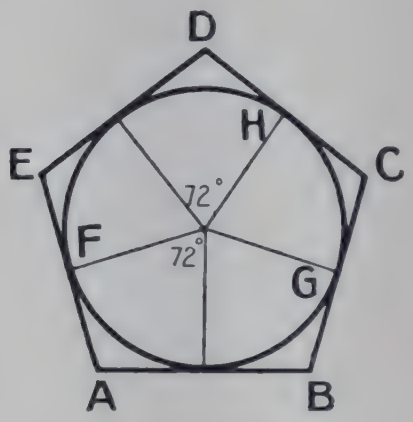
Find its centre F.

At F make the five ∠s AFB, BFC, CFD, . . . each = $\frac{4}{5}$ rt. ∠ = 72°.

Then the ◯ce ABCDE is divided into five equal parts, and by joining the pts. of division on the ◯ce, ABCDE is the required pentagon (III. 3; III. 4). Q.E.F.

Problem 13.

To circumscribe a regular pentagon about a given circle.



Let FGH be the given ⊙.

Find its centre F (III. 1).

Divide the ◯ce into five equal parts as in the previous problem.

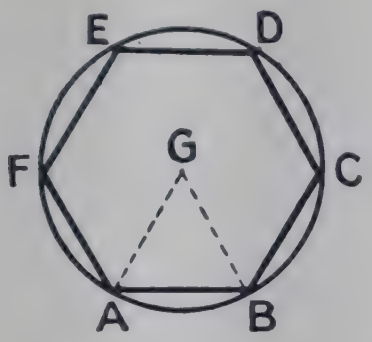
At the pts. of division on the ◯ce draw tans. AB, BC, CD, DE, EA to the ◯ce, cutting each other at the pts. A, B, C, D, and E.

Then ABCDE is the required pentagon.

Give proof.

Problem 14.

To inscribe a regular hexagon in a given circle.



Let **ABCDEF** be the given ⊙.

Find its centre **G**.

Place end to end in the ⊙, six chords **AB, BC, CD,** . . . each = radius **GA**.

Then **ABCDEF** is a regular hexagon.

Join **AG, GB**.

PROOF.—Now △ **AGB** is equilateral.

$\therefore \angle$ **AGB** $= 60°$;

$\therefore$ **AB** subtends an $\angle$ $60°$ at the centre.

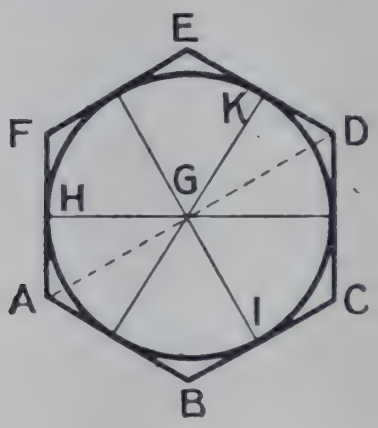
Sim$^{ly.}$ each chord subtends an $\angle$ $60°$ at the centre.

$\therefore$ the total $\angle$ subtended by the six chords $= 360°$.

$\therefore$ the six chords form a closed regular hexagon **ABCDEF**, Q.E.F.

Problem 15.

To circumscribe a regular hexagon about a given circle.



Let HIK be the given ⊙.

Find its centre G.

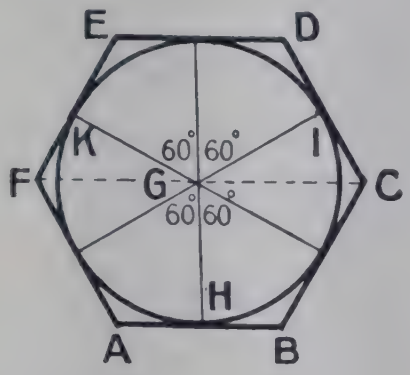
Take the radius GH within the legs of the compasses, and divide the ◯ce into six equal parts. At the pts. of division on the ◯ce draw the tans. AB, BC, CD, DE, EF, FA, cutting each other at the pts. B, C, D, E, F, A.

Then ABCDEF is the required hexagon.

Give proof.

Problem 16.

To circumscribe a regular hexagon about a given circle.



Let HIK be the given ⊙.

Find its centre G.

Divide the ⊙ into six equal parts by making the six ∠s at G each = 60°.

At the pts. of division on the ○ce draw the tans. AB, BC, CD, DE, EF, FA to the ⊙, cutting each other at the pts. A, B, C, D, E, F.

Then ABCDEF is the required hexagon.

Give proof.

General rule for inscribing a regular figure of three, four, five, six, or any number of sides in a given circle.

Find the centre of the circle. At the centre of it make the number of angles equal to the number of sides of the required figure, and each equal to 360° divided by the number of sides, *e.g.* for an equilateral triangle the three angles at the centre of the circle must each equal $\frac{360^\circ}{3} = 120^\circ$; for a square, the four angles at the centre must each equal $\frac{360^\circ}{4} = 90^\circ$, and so on. Then join the points of division on the circumference to form a closed figure within the given circle.

Exercise.

Inscribe a regular figure of (*a*) eight, (*b*) ten, (*c*) twelve sides in a given circle.

General rule for circumscribing a regular figure of three, four, five, six, or any number of sides about a given circle.

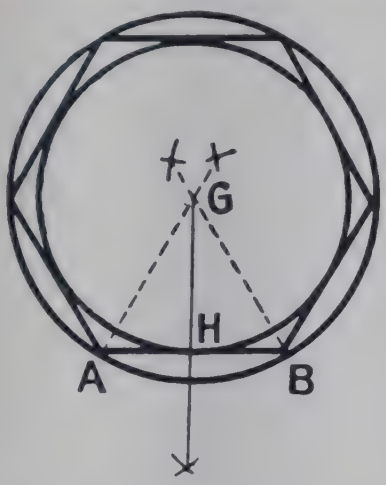
Find the centre of the circle, and divide the circumference, as in the previous paragraph, into the required number of equal parts, *i.e.* three for an equilateral triangle, four for a square, and etc. Then draw tangents to the circle at the points of division, and the required figure will be circumscribed.

Exercise.

Circumscribe a regular figure of (*a*) eight, (*b*) ten, (*c*) twelve sides about a given circle.

Problem 17.

(*a*) *To circumscribe a circle about a given regular polygon.*
(*b*) *To inscribe a circle in a given regular polygon.*



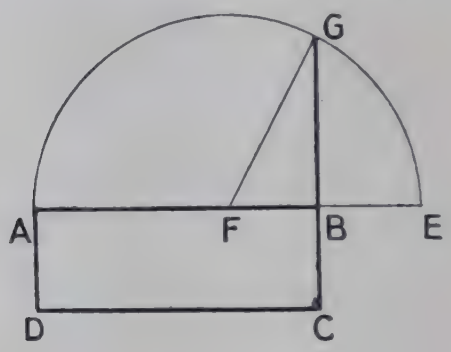
(*a*) Bisect any two adj. ∠s **A** and **B** by **AG, BG.**
With centre **G** and radius **GA** draw a ⊙.
This is the circumscribing ⊙.
(*b*) Draw **GH** ⊥ **AB**, meeting it in **H.**
With centre **G**, and radius **GH**, describe a ⊙.
This is the inscribed ⊙.

Exercise.

Apply the above methods to any given regular polygon.

Problem 18.

To construct a square equal in area to a given rectangle.



Let ABCD be the given rect.

Produce AB to E, making BE = BC.

Bisect AE at F. With centre F, and radius FE, describe the semi-⊙ AGE.

Produce CB to meet the semi-⊙ in G.

Then BG is a side of the required square.

PROOF.--Since AE is divided equally at F and unequally at E.

$$\therefore \text{ the rect. } AB \cdot BE + FB^2 = FE^2 \text{ (II. 6)}$$
$$= FG^2$$
$$= FB^2 + BG^2.$$

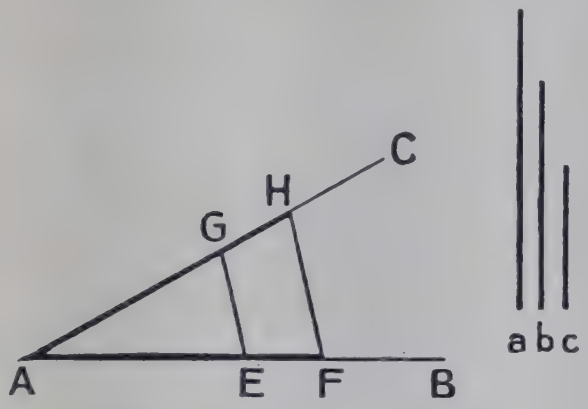
Taking away the common sq. FB^2 from both sides

rect. AB . BE = rect. AB . BC = BG^2. Q.E.F.

As a corollary it might be stated that if from any point in the circumference of a circle a perpendicular be drawn to meet the diameter, the square on the perpendicular is equal to the rectangle contained by the segments into which it divides the diameter.

Problem 19.

To find a fourth proportional to three given straight lines.



Let a, b, c be the three given strt. lines.

Draw any two strt. lines AB, AC, making a convenient $\angle$.

Along AB set off AE and AF $= b$ and a respectively. Along AC set off AG $= c$.

Join GE. Draw FH || EG, meeting AC in H.

Then GH is the required fourth proportional.

PROOF.— Since GE || FH,

$$\therefore \frac{AE}{AF} = \frac{AG}{GH}$$

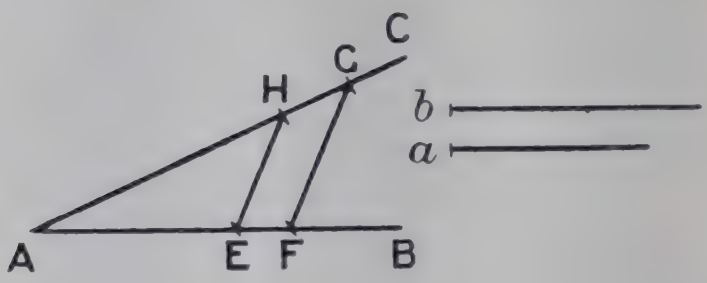
$$\text{i.e. } \frac{b}{a} = \frac{c}{AH}. \quad \text{Q.E.F.}$$

Exercise.

Find graphically the fourth proportional to 3, 4, 5. Check result by calculation.

Problem 20.

To find a third proportional to two given straight lines.



Let a, b be the two given strt. lines.

Draw any two strt. lines AB, AC, making a convenient $\angle$.

Along AB set off AE $= a$; along AB, AC set off AF, AH each $= b$.

Join HE; draw FG || EH, meeting AC in G.

Then AG is the required third proportional.

Give proof.

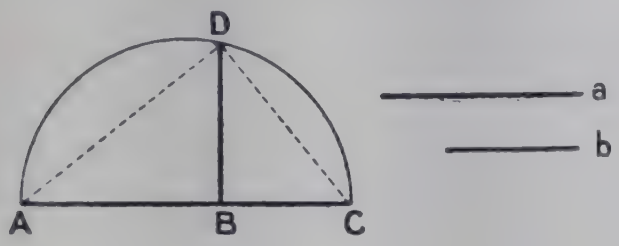
Exercises.

(1) Divide a given straight line internally and externally in a given ratio. Is this always possible?

(2) Find graphically the third proportional to 6·3, 8·6. Check result by calculation.

Problem 21.

To find a mean proportional between two given straight lines.



Let *a* and *b* be the two given strt. lines.

Place *a*, *b* in a strt. line as AB, BC, and on AC describe a semi-⊙ ADC.

From B draw BD ⊥ AC.

Then BD is the mean proportional required.

Join AD, DC.

PROOF.— Now ∠ ADC = rt. ∠ (III.),

and △s ABD, DBC are equiangular and ∴ similar;

$$\therefore \frac{AB}{BD} = \frac{BD}{DC}$$

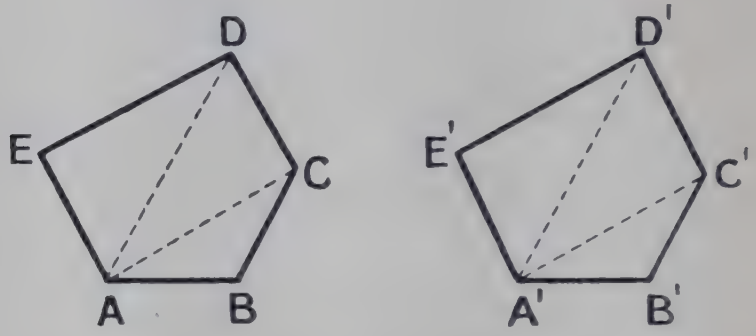
$$i.e.\ \frac{a}{BD} = \frac{BD}{b}. \quad \text{Q.E.F.}$$

Exercise.

Find graphically the mean proportional between (i.) 1 and 3, (ii.) 1 and 4.

Problem 22.

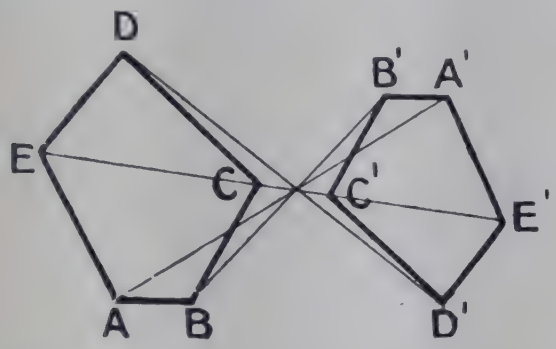
On a given straight line to construct a figure similar to a given rectilinear figure.



Let ABCDE be the given figure, and A′B′ the given strt. line.
Join AC, AD.
On A′B′ make △ A′B′C equiangular to △ ABC.
On A′C′ make △ A′C′D′ equiangular to △ ACD.
On A′D′ make △ A′D′E′ equiangular to △ ADE.
Then A′B′C′D′E′ is similar to ABCDE.
Using Prop. 2, Bk. IV., prove that the figures are similar.

Problem 23.

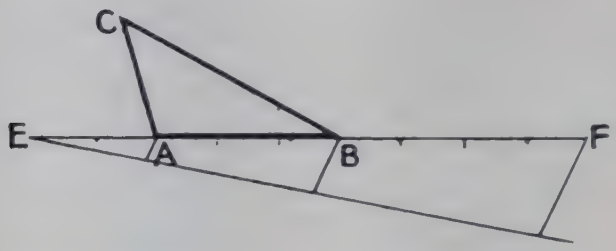
On a given straight line to construct a figure similar to a given rectilinear figure.



From the method indicated in the above diagram construct a figure similar to the given rectilinear figure ABCDE.

Problem 24.

To construct a triangle whose sides are in the ratio 2 : 3 : 4.



Draw a line EF and divide it into nine equal parts or units of length.

Make EA = two units of length.
„ AB = three „ „
„ BF = four „ „

On AB, as base, describe the △ ABC, making AC = AE, and BC = EF.

Then ABC is the required △. Q.E.F.

Exercise.

Construct a triangle of given perimeter, whose sides are in the ratio 3 : 4 : 5.

Miscellaneous Practical Exercises.

(1) Draw a circle to touch a given circle, and also to touch two given straight lines.

(2) Find a point in a given straight line, such that its distance from a given point is equal to its distance from the circumference of a given circle.

(3) Find graphically the fourth proportional of (*a*) 3, 4, 5; (*b*) 4·5, 6, 8. Check result by calculation.

(4) Find graphically a third proportional to 1·5 and 1.

(5) Show graphically that $\frac{6 \times 4}{2} = 12$.

(6) Find graphically the values of $\sqrt{2}$, $\sqrt{3}$. Correct to two places of decimals.

(7) Find graphically a mean proportional between—

(*a*) 1 and 10. (*b*) 4 and 9. (*c*) 3 and 6. (*d*) $\frac{1}{3}$ and 3.

(8) Divide a given straight line into three segments, in the ratio 3 : 5 : 7.

(9) Draw a triangle, and construct an equivalent rectangle.

(10) Describe squares equal in area to the following figures :—

(*a*) A rectangle with sides 3 in. and 4 in.

(*b*) A parallelogram with sides 1·5 in. and 2·5 in., and an included angle of 60°.

(*c*) A quadrilateral ABCD given (AB = 10 cm., BC = 8 cm., CD = 9 cm., DA = 7 cm., BD = 8 cm.).

(11) Describe equilateral triangles on the side and diagonal of a square. Determine the ratio of their areas.

(12) On the sides of a right-angled triangle describe equilateral triangles. Find the ratio of the areas of the three equilateral triangles.

(13) The areas of two similar triangles are 64 sq. in. and 36 sq. in. ; the base of the greater is 8 in. By construction and calculation determine the base of the other.

(14) Describe a figure to prove that equiangular polygons are not necessarily similar.

(15) Describe a triangle, having given its angles and the difference of two of its sides.

LIST OF DEFINITIONS

An **angle** is the inclination of two straight lines, which meet together, but are not in the same straight line.

Adjacent angles are angles having a common vertex, a common arm, and do not overlap.

Straight angle. If the two arms of an angle lie in the same straight line, then the angle formed by them on either side is called a straight angle.

Right angle, perpendicular. When a straight line standing on another straight line makes the adjacent angles equal to one another, each of the angles is a right angle. The straight line which stands on the other is called a perpendicular to it.

An **obtuse angle** is an angle greater than a right angle, but less than two right angles.

An **acute angle** is an angle which is less than a right angle.

Supplementary angles. If two angles be together equal to two right angles, they are said to be supplementary.

Complementary angles. If two angles be together equal to a right angle, they are said to be complementary.

Vertically opposite angles. When two straight lines intersect one another, the opposite angles so formed are termed vertically opposite angles.

A **reflex angle** is an angle greater than two right angles, and less than four right angles.

The **area** of a figure is the amount of surface enclosed by its boundary.

A **unit of area** is a square with a side of unit length.

Congruency. Figures are said to be congruent when they can be made to coincide.

A **circle** is a plane figure contained by one line, which is called the circumference, and is such that all the straight lines drawn from a certain point within the figure to the circumference are equal to one another. This point is called the centre of the circle.

The **radius** of a circle is a straight line drawn from the centre to the circumference.

The **diameter** of a circle is a straight line drawn through the centre and terminated both ways by the circumference.

A **semicircle** is the figure bounded by the diameter and part of the circumference cut off by the diameter.

A **segment** of a circle is the figure bounded by any straight line and part of the circumference which it cuts off.

A **chord** of a circle is a straight line joining any two points on the circumference.

An **arc** of a circle is any part of the circumference.

A **secant** of a circle is a chord produced both ways.

The **angle in a segment** of a circle is the angle subtended by the chord of the segment at a point on the arc.

Contact of circles. Circles are said to touch one another at a point when they have a common tangent at that point, or when their two points of intersection coincide.

Circumscribed circle. A circle is said to be circumscribed about a rectilineal figure when the circumference of the circle passes through all the angular points of the figure.

A rectilineal figure is said to be **circumscribed** about a circle when each side of the figure touches the circle.

Inscribed circle. A circle is said to be inscribed in a rectilineal figure when the circumference of the circle is touched by each side of the figure.

Concyclic. Four or more points which lie on the same circle are said to be concyclic.

Cyclic. A rectilineal figure which is such that a circle can be drawn about it is said to be cyclic.

Line (straight). A straight line is one such that only one of its kind can pass through two given points.

Line (curved). If a line be not straight, it is said to be curved. A curved line is one of varying direction.

Locus. If a line, or group of lines, satisfy a certain geometrical condition, and if all the points satisfying the given condition be on that line or group of lines, then the line or group of lines is called the locus of a point.

Collinear. Three or more points are said to be collinear when they lie on the same straight line.

Parallel straight lines are such as, being in the same plane, do not meet, however far they are produced in either direction.

A **polygon** is a plane figure bounded by four or more straight lines.

A **convex polygon** is one whose boundary cannot be intersected at more than two points by a straight line drawn across the figure.

A **pentagon, hexagon, heptagon, octagon, nonagon, decagon** is a polygon of five, six, seven, eight, nine, ten sides.

A **pentagram** is a five-sided, star-shaped figure in which each side is met by all the other sides without the necessity of producing any of the sides.

A **hexagram** is a six-sided figure in which any side is met by the opposite side, *i.e.* fourth in order counting from that side, without the necessity of producing any side.

A **quadrilateral** is a plane figure bounded by four straight lines.

A **diagonal** of a quadrilateral is the straight line joining the opposite angular points.

A **parallelogram** is a four-sided figure consisting of two pairs of parallel sides.

A **rectangle** is a parallelogram which has one of its angles a right angle.

A **square** is a four-sided figure which has all its sides equal, and one of its angles a right angle.

A **rhombus** is a four-sided figure which has all its sides equal, but its angles are not right angles.

A **rhomboid** is a four-sided figure consisting of two pairs of parallel sides, but its angles are not right angles.

A **trapezium** is a four-sided figure which has two of its sides parallel.

Trapezium isosceles and **scalene**. If the trapezium has the sides which are not parallel equal to one another, it is an isosceles trapezium ; if the four sides be uneven it is a scalene trapezium.

Surface is that which has length and breadth, but not thickness.

A **plane surface** is one such that only one of its kind can pass through three given points.

A **triangle** is a plane figure bounded by three straight lines.

An **obtuse-angled** triangle is a triangle which has an obtuse angle.

A **right-angled** triangle is a triangle which has a right angle.

An **acute-angled** triangle is a triangle which has three acute angles.

An **equilateral** triangle is a triangle which has three equal sides.

An **isosceles** triangle is a triangle which has two of its sides equal.

A **scalene** triangle is a triangle which has three unequal sides.

Any side of a triangle may be taken as the **base,** and then the corresponding vertex is the opposite angular point.

A **median** of a triangle is the straight line joining an angular point to the middle point of the opposite side.

POSTULATES.

Let it be granted—

(1) That a straight line may be drawn from any one point to any other point.

(2) That a finite, that is to say, a terminated straight line may be produced to any length in that straight line.

(3) That a circle may be described from any centre, at any distance from that centre, that is, with a radius equal to any finite straight line drawn from the centre.

GENERAL AXIOMS.

(1) Things which are equal to the same thing are equal to one another.

(2) If equals be added to equals, the wholes are equal.

(3) If equals be taken from equals, the remainders are equal.

(4) If equals be taken from unequals, the remainders are unequal, the greater remainder being that which is left from the greater of the unequals.

(6) Things which are double of the same thing, or of equal things, are equal to one another.

(7) Things which are halves of the same thing, or of equal things, are equal to one another.

(8) The whole is greater than its part.

GEOMETRICAL AXIOMS.

(9) Magnitudes which can be made to coincide with one another are equal.

(10) Two straight lines cannot enclose a space.

(11) All right angles are equal.

OXFORD LOCAL EXAMINATIONS, 1905

SENIOR CANDIDATES.

WEDNESDAY, JULY 19, FROM 10.45 A.M. TO 12.15 P.M.

Geometry.

The constructions in questions 1, 3, *and* 6 *should be explained, and shown to be correct.*

1. Draw a circle through three points which are not in the same straight line.

2. If the square on one side of a triangle is equal to the sum of the squares on the other two sides, prove that the angle contained by these two sides is a right angle.

3. Divide a straight line into two parts so that the square on one part may be double the square on the other part.

4. Prove that the angle in a semicircle is a right angle, and that the angle in a segment less than a semicircle is greater than a right angle.

5. If P is a fixed point outside a circle, and a straight line through P meets the circumference of the circle in Q and R, prove that the rectangle PQ . PR is equal to the square on the tangent drawn to the circle from P.

Find the locus of the mid-point of QR.

6. Construct a triangle having given the base, the vertical angle, and the sum of the remaining sides.

7. If ABCD is a square circumscribing a circle, and any tangent to the circle meets AB, BC, CD, AD in P, Q, R, and S respectively, prove that PQ, QR, RS subtend equal or supplementary angles at the centre of the circle.

8. Four points A, B, C, D are taken on a circle so that the chords AC, BD are at right angles to each other. Prove that the tangents at A, B, C, D form a quadrilateral which can be inscribed in a circle.

9. If a radius of a circle is drawn perpendicular to a given diameter, and through the extremity of the radius any straight line is drawn, prove that the sum of the squares on the perpendiculars drawn to this straight line from the extremities of the diameter is constant, and equal to twice the square on the radius of the circle.

SATURDAY, JULY 22, FROM 12 TO 1 P.M.

Geometrical Drawing.

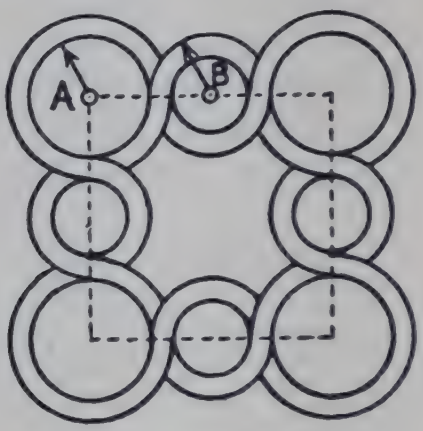
The construction must in every case be fully shown.

1. A quadrilateral figure has two equal adjacent sides 2″ in length which include an angle of 45°. The other sides are equal and include an angle of 90°. Inscribe a semicircle within it, the longer sides being tangents to the arc.

2. Construct an isosceles triangle ABC having base AB $1\frac{1}{2}$" and vertical angle 30°. Mark a point D 2" above B. Make a triangle equal in area to ABC having its base in AB and its apex at D.

3. The given pattern is drawn to a scale of $\frac{3}{4}$" to a foot. It is required to be redrawn to a scale of 1" to a foot. The radii marked A and B are equal.

Note.—The construction lines should be drawn, but not more than half the pattern need be finished.



JUNIOR CANDIDATES.

WEDNESDAY, JULY 19, FROM 10.45 A.M. TO 12.15 P.M.

Geometry.

Marks will be awarded for neatness in the drawing of figures.

In the constructions in questions 1 *and* 2, *no explanations need be given, but all lines required in those constructions must be clearly shown.*

1. Construct a rhombus which shall have an area of 96 square centimetres, and each side 10 centimetres in length. Measure the longer diagonal in centimetres.

2. Draw a triangle with sides 2·5, 6, and 6·5" long. Inscribe a circle in the triangle, and measure its radius in inches.

3. If two angles of a triangle are equal, prove that the sides opposite to them are also equal.

4. Prove that the diagonals of a parallelogram bisect one another.

5. If the radius of a circle is 5 centimetres long, find the locus of the mid-points of all chords of the circle which have a length of 6 centimetres.

6. Prove that the opposite angles of any quadrilateral inscribed in a circle are supplementary.

7. Construct a triangle ABC, having given the lengths of the perpendicular from B to AC, and of the sides AB, AC. Explain your construction and show that it is correct.

8. If two circles touch one another externally at P, and a common tangent to the circles touches one circle at Q and the other at R, prove that QR subtends a right angle at P.

FRIDAY, JULY 21, FROM 11.15 A.M. TO 12.45 P.M.

Higher Geometry.

Write Higher Geometry at the head of each sheet of your answers, or you will obtain no credit for your work.

Marks will be awarded for neatness in the drawing of figures.

In the constructions in questions 1 *and* 2, *no explanations need be given, but all lines required in those constructions must be clearly shown.*

1. Draw two circles, one with radius 2 centimetres and the other with radius 3 centimetres, having their centres 10 centimetres apart, and construct a third circle with radius 4 centimetres touching each of them.

2. Divide a straight line AB, 6 centimetres long, internally at X and externally at Y, in the ratio of 5 : 2. Measure AY in centimetres.

3. ABC and DEF are similar triangles having AB and DE corresponding sides. If the area of DEF is four-ninths of the area of ABC, and DE is 4″ long, calculate the length of AB.

4. If two circles intersect, prove that the straight line which joins their centres bisects their common chord at right angles.

What does this theorem become, in the limit, when the circles touch one another?

SATURDAY, JULY 22, FROM 12 TO 1 P.M.

Geometrical Drawing.

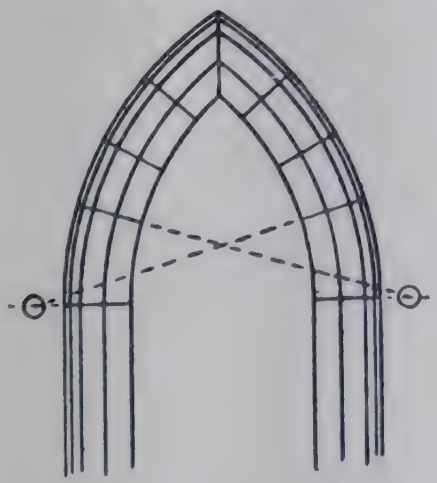
Both neatness and accuracy of work will be required to obtain full marks for any question. Candidates are advised not to ink in their problems.

1. Find a third proportional to two lines $1\frac{1}{2}$″ and $2\frac{1}{4}$″ long. Upon this line construct an equilateral triangle.

2. Within a circle of $1\frac{1}{2}$″ radius inscribe two other circles of $\frac{1}{2}$″ and $\frac{3}{4}$″ radius respectively, to touch each other and the circumference of the given circle.

3. Construct a scale $2\frac{1}{2}$″ to 1 yd. to show $5\frac{1}{2}$′. The scale must be neatly figured, and read to divisions of 6″.

4. Make a copy of the diagram of the Lancet-Arch of double the size.



PRELIMINARY EXAMINATION.

WEDNESDAY, JULY 19, FROM 10.45 A.M. TO 12.

Geometry.

Marks will be awarded for neatness in the drawing of figures.

In the constructions numbered 1, 5, 6 *no explanations need be given, but all lines required in the constructions must be clearly shown.*

1. Draw a straight line 4·7″ long and measure its length in centimetres.

2. Bisect a given angle. Explain your construction and prove that it is correct.

3. If two straight lines intersect, prove that the vertically opposite angles are equal.

4. Prove that the straight lines which join the extremities of two equal and parallel straight lines towards the same parts are themselves equal and parallel.

5. Construct a triangle ABC in which AB = 3″, BC = 4″, and the angle A = 72°.

Measure the side AC.

6. Construct a parallelogram ABCD having

AB = 6 centimetres,
AD = 10 centimetres,
and AC = 14 centimetres.

Measure the angle ABC.

PRELIMINARY EXAMINATION.

THURSDAY, JULY 20, FROM 9 TO 10 A.M.

Higher Geometry.

Marks will be awarded for neatness in the drawing of figures.

In the construction in question 1 *no explanation need be given, but all lines required in the construction must be clearly shown.*

1. A chord of a circle is 6″ long, and its perpendicular distance from the centre of the circle is 4″. Construct the circle and determine its radius.

2. Prove that the greater of two unequal chords of a circle is nearer to the centre of the circle than the less.

3. Draw the tangents to a circle from an external point P. Explain your construction and prove that it is correct. If the circle has a radius of 5 centimetres, and the distance of P from the centre is 13 centimetres, *calculate* the length of each tangent.

4. If the interior of a cylindrical box is 10″ high and 2″ in diameter, how many spheres of 1″ radius can be placed inside it?

5. Find the locus of the centres of all circles which pass through two given points.

WEDNESDAY, JULY 19, FROM 2 TO 3 P.M.

Geometrical Drawing.

1. Construct a plain scale of $\frac{1}{10}$ to measure yards and feet. The scale must be long enough to measure 5 yards.

2. A triangle ABC has the base AC = 2″, the angle BAC = 30°, and the angle ACB = 120°.

(*a*) Construct the triangle.

(*b*) From B draw a line in a direction perpendicular to the base AC without producing the base.

3. Describe a circle 3″ in diameter, divide the circumference into 5 equal parts, and draw the figure formed by joining the alternate points of division.

4. In a square of $3\frac{1}{2}$″ side inscribe a regular octagon.

5. From a point A draw two straight lines AB and AC inclined to each other at an angle of 45°, and describe a circle of 1″ radius touching both lines. Draw a second circle in the space between the first circle and the point A touching the lines AB and AC and also the circle.

1906.

SENIOR CANDIDATES.

TUESDAY, JULY 17, FROM 11.15 A.M. TO 12.45 P.M.

Geometry.

The constructions in questions 1, 4, *and* 6 *should be explained, and proved to be correct.*

1. Prove that triangles on the same base and between the same parallels are equal in area.

From a point on the base of a triangle draw a straight line to divide the area of the triangle into two equal parts.

2. If the internal bisector of the angle A of a triangle ABC meets the perpendicular bisector of the base BC in the point O, and if perpendiculars OM, ON to the sides AB, AC meet them respectively in M and N, prove that

(1) AM = AN ; (2) BM = NC.

3. Illustrate geometrically the identity

$$(a+b)(a-b) = a^2 - b^2.$$

If the diagonal AC of a rhombus ABCD is produced to any point P, prove that

$$PA \cdot PC = PB^2 - AB^2.$$

4. Divide a straight line into two parts so that the rectangle contained by the whole line and one of the parts is equal to the square on the other part.

5. Prove that the angles in the same segment of a circle are equal.

Two circles intersect at A and B ; through A two lines LM and PQ are drawn, meeting the first circle in L and P, and the second circle in M and Q. If these four points are joined to B, prove that the angles LBP, MBQ are equal.

6. Define a tangent to a circle, and prove that it is at right angles to the radius drawn to the point of contact.

Draw a circle to pass through a given point P, and to touch a given circle at a given point Q. When does the construction appear to fail ?

7. Chords of a circle are drawn through a given internal point O; prove that the rectangle under their segments is constant.

8. Tangents are drawn at the ends of a fixed diameter AB of a circle, and are cut by a third tangent in points P, Q respectively; prove that the rectangle AP . BQ is constant.

JUNIOR CANDIDATES.

TUESDAY, JULY 17, FROM 11.15 A.M. TO 12.45 P.M.

Geometry.

Marks will be awarded for neatness and accuracy in the drawing of figures.

In the constructions in questions 2 *and* 3 *no explanation need be given, but all lines required in those constructions must be clearly shown.*

1. Prove that, if, in two triangles ABC, PQR, the sides AB, BC are respectively equal to the two sides PQ, QR, and the angle ABC is equal to the angle PQR, the two triangles are equal in all respects.

Prove that the triangle formed by joining the mid-points of the sides of an isosceles triangle is also isosceles.

2. Prove that triangles on the same base and between the same parallels are equal in area.

On a base of 2″ describe a triangle having an area of $1\frac{1}{2}$ sq. in., and having one side 3″ long.

3. Prove that, in a right-angled triangle, the square on the hypotenuse is equal to the sum of the squares on the other two sides.

If the hypotenuse measures 39′ and one of the sides measures 15′, calculate the length of the other side, and verify by drawing a triangle to the scale 1 cm. = 10′.

4. Two equal circles intersect at A and B; and through B a straight line is drawn cutting the circles again in P and Q. Prove that P and Q are equidistant from A.

5. Show how to inscribe a circle in a given triangle, proving the correctness of the method.

Draw two diameters of a given circle, crossing each other at right angles, and inscribe circles in the four sectors so formed. (Your construction must be explained and its accuracy proved.)

6. Define—a right circular cone, a prism, a pyramid.

In a triangular pyramid how many edges are there, and how many corners?

JUNIOR CANDIDATES.

THURSDAY, JULY 19, FROM 9 TO 10 A.M.

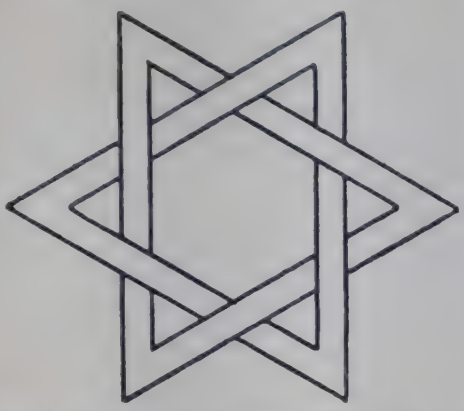
Geometrical Drawing.

Both neatness and accuracy of work will be required to obtain full marks for any question. Candidates are advised not to ink in their problems.

1. Draw a line 5″ in length to represent 300′. Construct upon this line a diagonal scale to show single feet. Figure your scale neatly.

2. Construct a triangle on a base of $2\frac{1}{2}$″, the vertical angle to be 60° and one of the remaining sides 2″. About this triangle describe a circle.

3. Construct a rectangle on a diagonal of 3″, two sides being each $1\frac{1}{2}$″ long.



4. Make a copy of the above diagram of twice the size. (The construction is based on a regular hexagon.)

PRELIMINARY EXAMINATION.

WEDNESDAY, JULY 18, FROM 9 TO 10.15 A.M.

Geometry.

Marks will be awarded for neat and accurate figures.
Lines and arcs used in constructions should not be rubbed out.

1. Draw a triangle whose sides are 10 cm., 7·5 cm., and 5·5 cm. Measure its angles.

Draw the perpendicular from the greatest angle to the opposite side of this triangle without using the set-square or protractor; and measure the lengths of the parts into which this side is divided by the perpendicular.

2. Show how to bisect an angle, and prove the correctness of your method.

Draw two straight lines AB, CD cutting each other at a point O, so that the angle AOC = 50°. Bisect the angles AOC and COB by lines OK and OL, and find by calculation the number of degrees in the angle KOL.

3. If two triangles ABC, PQR have their sides equal, namely, AB = PQ, BC = QR, and CA = RP, prove that the triangles are equal in every respect.

4. Prove that parallelograms on the same base and between the same parallels are equal in area.

Draw a parallelogram with sides $2\frac{1}{2}$″ and $1\frac{1}{2}$″, having an area of 3 sq. inches, and explain your method of obtaining the correct area.

PRELIMINARY EXAMINATION.

THURSDAY, JULY 19, FROM 9 TO 10 A.M.

Higher Geometry.

Marks will be awarded for neatness and accuracy in the drawing of geometrical figures.

1. Draw a quadrilateral figure ABCD such that AB = AD, and CB = CD. Prove that the diagonal AC bisects the angles at A and C.

Prove also that the diagonals AC, BD cut each other at right angles.

2. If in a triangle ABC, the square on BC equals the sum of the squares on AB, AC, prove that the angle BAC is a right angle.

Construct a line whose length $= \sqrt{15}$ centimetres.

3. Prove that an angle at the centre of a circle is double an angle at the circumference standing on the same arc.

Draw a figure showing an angle of 120° at the circumference ; and state carefully which is the corresponding angle at the centre. What fraction of the whole circumference is the arc on which the angle stands ?

4. Show how to draw a common tangent to two circles, and to find its two points of contact.

Draw a tangent to two circles whose centres are 6·5 cm. apart and whose radii are 5 and 2·5 cm. ; and calculate the length of this tangent between the points of contact.

5. Show that the square on a diagonal of a cube is equal to three times the square on any edge.

PRELIMINARY EXAMINATION.

THURSDAY, JULY 19, FROM 10 TO 11 A.M.

Geometrical Drawing.

1. (*a*) Construct a plain scale to represent miles and furlongs, taking $\frac{1}{12240}$, as the representative fraction and showing at least four miles.

(*b*) What line in your figure represents two miles three furlongs ?

2. Describe five circles, equal to and touching each other, surrounding and touching a circle of one inch diameter.

3. Construct a triangle with sides in the ratio 5 : 6 : 7, the longest side being 4″ in length. In this triangle inscribe a circle.

4. Inscribe a regular heptagon in a circle of $2\frac{1}{2}$″ diameter.

5. Construct a square having each diagonal 1″ longer than a side.

6. In a quadrilateral ABCD

the side AB = 1″
,, AD = 2″
the angle BAD = 120°.

The two remaining sides BC and CD are equal to each other, and the angle BCD, contained by them, is 60°. Construct the figure.

CAMBRIDGE EXAMINATION PAPERS, 1905

PRELIMINARY.

THURSDAY, DECEMBER 14, 1905. 10 TO 12 OR 10½ TO 12½.

Geometry.

Candidates can pass in Geometry by doing sufficiently well in Part I. Figures should be drawn accurately with a hard pencil. In the constructions numbered 1, 2, 6, 7, 8, *no explanations need be given;* but all lines required in the constructions must be shown clearly.

PART I.

1. Draw two straight lines through a point O including an angle of 49°; take a point P in one of them at a distance of 5·7 centimetres from O, and draw PM perpendicular to the other line. Measure the length of OM. [The set square must not be used in answering this question.]

2. Construct a triangle having its sides equal to 3·6″, 2·8″, and 1·3″ respectively, and measure the greatest angle.

3. What are the lengths of the edges of a rectangular box which will just hold two cylinders lying side by side, the length of each cylinder being 6 inches and the radius of the ends 1″?

4. Prove that the angles at the base of an isosceles triangle are equal.

5. If, in two triangles, the three sides of one are equal to the three sides of the other respectively, show that the triangles are equal in all respects.

ABC is an isosceles triangle having AB = AC, and the exterior angles at B, C are bisected by lines meeting in D; prove that DA bisects the angle BAC.

6. Prove that the diagonals of a parallelogram bisect one another.

Draw a parallelogram whose diagonals are of lengths 4·5 centimetres and 9·3 centimetres, and make with each other an angle of 62°; also find the area of the parallelogram.

PART II.

7. Draw a circle of radius 0·9″; draw a line through the centre; on it take two points each 2·3″ from the centre, and draw the tangents from these points. What is the figure formed by the four tangents? Verify by suitable measurements.

8. Draw a triangle ABC, in which AB = 5·2 centimetres, AC = 8·1 centimetres, and the angle A = 41°. Describe a circle round the triangle and measure its radius.

9. Show that equal chords of a circle are equally distant from the centre.

A chord of a circle is of constant length, but its position changes; a point is taken on it at a fixed distance from one of its ends. Prove that this point always lies on a fixed concentric circle.

10. Show that the angle subtended by a diameter of a circle at any point of its circumference is a right angle.

AD, BE, CF are the perpendiculars from the vertices of a triangle ABC upon its opposite sides; show that the angles BAD, FEB, DEB, BCF are all equal.

THURSDAY, DECEMBER 14, 1905. $3\frac{3}{4}$ TO $4\frac{1}{4}$ OR $4\frac{1}{4}$ TO 5.

Geometrical Drawing.

[*Mathematical Instruments may be used.*]

[The solutions must be numbered to correspond with the problems, and the lines by which each problem is solved must be shown. The exercises are to be worked in pencil only, and marks will be awarded for neatness and precision of execution.]

Candidates will find it convenient to draw two straight lines, horizontal and vertical, dividing each sheet of paper into four equal spaces, as shown in the diagram. A figure can be placed in each division.

Two sheets of paper will be given to each candidate.

1. Construct a triangle two of the sides of which are respectively $3\frac{1}{8}''$ and $2\frac{3}{4}''$ in length, the contained angle being 60°. Draw perpendiculars from each of the corners of the triangle to the opposite sides.

2. Draw a square on a straight line $2\frac{7}{8}''$ in length, and circumscribe it by a circle.

3. Draw a circle of radius $1\frac{5}{8}''$, having its centre at O. Place within it a chord AB $2\frac{1}{2}''$ in length. Construct at A an angle BAC equal to the angle BOA.

4. Draw a plain scale of $1\frac{1}{4}''$ to 1′ to show feet and inches, 4′ being the greatest distance shown.

5. A, B, C, D, E, F are the corners of a regular hexagon, each side of which is 11 miles. Construct the hexagon to the scale of 1″ to 10 miles, and measure the lengths of AC and AD to the nearest mile.

[*The scale need not be drawn.*]

6. Construct an equilateral triangle on a base of 2″. Circumscribe it by a circle, and inscribe a circle within it.

[*This problem is to be a test of accurate drawing.*]

CAMBRIDGE JUNIOR, 1905.

FRIDAY, DECEMBER 15, 1905. 9 TO 11½ OR 9½ TO 12.

Geometry.

Candidates can pass in Geometry by doing sufficiently well in the A *part of this paper. Figures should be drawn accurately with a hard pencil. In questions* 1, 2, 7, *and* 8, all lines required must be shown clearly.

Answers to the questions marked A *are to be fastened together in one bundle; those to the questions marked* B *in another bundle.*

A.

1. Draw a quadrilateral ABCD in which AB = CD = 7 centimetres, AD = BC = 5 centimetres, and the diagonal AC = 11 centimetres. Measure the angles, and use your results to verify the fact that the figure is a parallelogram.

Prove also by geometrical reasoning that the figure is a parallelogram.

2. Draw a triangle ABC in which AB = 3·6″, AC = 2·7″, ∠ A = 53°. Find the two points P, Q, each of which is equidistant from B and C, and is also equidistant from BA and CA (produced if necessary). Measure the length of PQ. State exactly the various steps of your construction.

3. Prove that if ABC, DEF are two triangles in which AB = DE, AC = DF, and ∠A = ∠D, then the triangles are equal in every respect.

In a triangle PQR the sides PQ, PR are equal; any point X is taken in QR; a point Y is taken in RP such that RY = QX, and Z is taken in PQ such that QZ = RX. Prove that if XY, XZ are joined, then XY = XZ.

4. Prove that the common chord of two intersecting circles is bisected at right angles by the line joining the centres of the circles.

Deduce (or prove in any way) that, if two circles touch externally, the line joining their centres passes through the point of contact and is perpendicular to the tangent at the point of contact.

5. Prove that angles in the same segment of a circle are equal.

ABC is any triangle, and AP, BQ, CR are perpendiculars to BC, CA, AB respectively. Show that PA bisects the angle QPR.

6. Prove that parallelograms on the same base and of the same altitude contain equal areas.

Draw a quadrilateral ABCD in which AB = 3·2″, BC = 3·4″, CD = 1·6″, DA = 2·9″, AC = 3·8″. Find the area of ABCD correctly to within one-tenth of a square inch.

B.

7. Draw a triangle in which AB = 7 centimetres, BC = 10 centimetres, CA = 8 centimetres; with centre A and with radius 3 centimetres, draw a circle touching the given circle *externally* and also touching BC at C. State exactly the various steps of your construction, and prove its correctness.

8. Draw a triangle ABC in which AB = 4 centimetres, AC = 5 centimetres, ∠A = 65°; and construct a square DEFG such that its vertices D, E lie on BC, F on CA, and G on AB. State exactly the various steps of your construction.

9. Prove that if two triangles are equiangular, then their corresponding sides are proportional.

A spherical ball of radius 5″ rests on a table; find the height of the smallest hollow cone, the diameter of whose base is 15″, which when placed on the table will cover the ball.

10. Prove that if ABCD is a quadrilateral inscribed in a circle, then

$$AB \cdot CD + BC \cdot DA = AC \cdot BD.$$

KLMN is a parallelogram, and the circle through KLM cuts KN at P. Prove that the difference between the squares on KL and KM is equal to the rectangle KN . KP.

11. Prove that the straight lines drawn from the vertices of a triangle perpendicular to the opposite sides are concurrent.

FRIDAY, DECEMBER 15, 1905. $6\frac{1}{2}$ TO $7\frac{1}{2}$.

Geometrical Drawing.

[*Mathematical instruments may be used.*]

[*The solutions must be numbered to correspond with the problems, and the lines by which each problem is solved must be shown. The words "protractor," or "set square," must be written against those problems for which these instruments have been used. Marks are awarded for precision of drawing.*]

Candidates will find it convenient to draw two straight lines, horizontal and vertical, dividing each sheet of paper into four equal spaces, as shown in the diagram. A figure can be placed in each division.

Two sheets of paper will be given to each candidate.

1. Find by geometrical construction a third proportional to 2″ and $2\frac{1}{4}$″. On this third proportional describe a triangle, the other sides of which are respectively 2″ and $2\frac{1}{4}$″ in length. Measure and write down the number of degrees in the greatest angle of the triangle.

2. Describe a rhombus, the longer diagonal of which is 4″, and whose acute angles are each 45°. Find a point P, within the figure at a distance of $1\frac{1}{4}$″ from each of two of the corners. Measure and write down the distances of P from the other two corners of the rhombus.

3. Describe a regular hexagon on a base of 1″. Construct a second hexagon concentric with and having its sides parallel to those of the first, such that the distance between two of its opposite sides is $2\frac{3}{8}$″.

4. Draw a straight line of length $4\frac{1}{4}$″, to represent 40 miles. Construct on this line a plain scale showing single miles, to measure up to 40 miles.

On a map two places B and C are found to be respectively 25 miles and 30 miles from A, and the angle BAC = 40°. By use of the scale find the distance from B to C, to the nearest mile.

5. Draw a horizontal straight line AB, 4″ in length. Mark upon it two points, S and H, so that AS = BH = 1″.

Draw an ellipse having S and H for its foci and AB for its major axis.

[*Only one quarter of the curve need be drawn.*]

6. Describe an equilateral triangle ABC, the side of which is 1·5″ long. Find D, E, F, the middle points of BC, CA, AB. With centres A, B, C respectively, draw circular arcs, with radii AF, BD, CE, to form a trefoil external to the triangle ABC. Draw a circle to circumscribe the figure.

[*This problem is to be a test of accurate drawing.*]

CAMBRIDGE SENIOR, 1905.

FRIDAY, DECEMBER 15, 1905. 9 TO 11 OR 9½ TO 11½.

Geometry.

N.B.—*All construction lines should be clearly shown. Attention is called to the alternative questions A, B, C, D at the end of the paper.*

1. Construct carefully, without using your set square or protractor, an isosceles right-angled triangle whose hypotenuse is 6 centimetres long.

On the hypotenuse of the triangle and on the same side of it as the opposite vertex describe a square. Use the figure to prove that the square on the hypotenuse is twice the square on either of the sides.

2. Describe a triangle the lengths of whose sides are 8 centimetres and 5 centimetres, and whose base is 10 centimetres in length. (i) Measure its angles with your protractor, (ii) measure its altitude, and (iii) calculate the area of the triangle, as accurately as your measurements permit.

State whether the triangle is acute or obtuse, and explain how you might have determined this without drawing the figure.

3. Determine (i) by geometrical construction and measurement, (ii) by arithmetic, the length of the fourth proportional to the lines whose lengths, in order, are 8 centimetres, 5 centimetres, and 10 centimetres, respectively.

4. Prove that the diagonals of a parallelogram bisect one another.

Criticize the following proof of the theorem that the diagonals AC, BD of a rhombus ABCD bisect one another at their point of intersection O: "In the triangles AOB, BCO we have AB equal to BC, and therefore the angle BAO is equal to the angle BCO, also BO is common to the triangles AOB, BCO, therefore the bases AO and OC are equal."

5. Given a circle whose centre is O, and an external point T, explain how the tangents from T to the circle can be constructed, and prove the correctness of your construction.

If the distance of T from O be twice the radius, find the angle between the tangents.

6. AOB, COD are two intersecting chords of a circle. State and prove the theorem which enables us to find the length of any one of the four segments AO, OB, CO, OD, when the lengths of the remaining three are known.

AB is a chord of a circle, and O is its middle point. C is one of the points in which the perpendicular through O to AB meets the circle. If the lengths of AB and OC are two feet and six inches respectively, find the radius of the circle.

7. Prove that the internal bisector of an angle of a triangle divides the opposite side internally in the ratio of the sides containing the angle, and likewise the external bisector externally.

Find the locus of a point which moves in such a way that its distances from two fixed points are in a given ratio.

8. What is meant by the angle between a straight line and a plane?

Two straight lines are given not lying in the same plane. Show how to draw a straight line which shall meet each of them at right angles.

9. Four equal spheres, each two inches in diameter, rest in contact with one another on a horizontal plane in such a way that their centres are the corners of a square, and a fifth equal sphere rests symmetrically on the four others. Find the distance from the plane of the centre of this last sphere.

N.B.—*One or more of the following questions A, B, C, D may be taken instead of an equal number of questions* 1–9, *but lower marks will usually be assigned to them.*

A. Prove that triangles on equal bases having the same altitude are equal in area.

Prove that if ABC is an equilateral triangle and O is the centre of its circum-circle, then the equilateral triangle described on any of the lines OA, OB, OC as base has an area equal to one-third of that of the triangle ABC.

B. Prove that the sum of the angles of a triangle is equal to two right angles.

Show that in an isosceles triangle the base is greater or less than either of the equal sides according as the vertical angle is greater or less than 60°.

C. Find the locus of points whose distances from two given points are equal.

Discuss the problem of finding a point on a given circle whose distances from two given points are equal.

D. Prove that angles in the same segment of a circle are equal to one another.

THURSDAY, DECEMBER 13, 1906. 10 TO 12 OR 10½ TO 12½.

Geometry.

Candidates can pass in Geometry by doing sufficiently well in Part I. Figures should be drawn accurately with a hard pencil. In the constructions numbered 1, 2, 6, 7, 8, *no explanations need be given;* but all lines required in the constructions must be shown clearly.

PART I.

1. Draw a straight line AB, 2·5″ long; also a straight line BC, 1·4″ long, perpendicular to AB. Measure the angle BAC. [The set square must not be used.]

2. In the triangle ABC, AB = 6·5 centimetres, AC = 9·7 centimetres, and the angle A = 55°. Measure BC, and the angle B.

3. How many cubes with an edge of 3″ can be made out of a cube with an edge of 12″?

4. If two angles of a triangle are equal, prove that the triangle is isosceles.

PQR is an isosceles triangle in which the angles PQR, PRQ are equal. These equal angles are bisected by the straight lines QO, RO which meet at O. Show that the triangle OQR is isosceles.

5. In the triangles ABC, DEF the side BC is equal to the side EF; also the angles at B, C are equal to the angles at E, F respectively. Show that the triangles are equal to each other in all respects.

Straight lines AD, AE meet the base BC of a triangle ABC at D, E, and the angle BAD is equal to the angle CAE. If AD is equal to AE, prove that the triangle ABC is isosceles.

6. Show that parallelograms on the same base and between the same parallels are equal in area.

Draw a rectangle whose diagonals are each 9 centimetres long and make with each other an angle of 30°. Find its area, and show that it is half that of a square whose diagonal is 9 centimetres.

PART II.

7. Measure, along a straight line OAB, lengths OA, OB of 7·5 and 9 centimetres respectively. With A and B as centres draw circles whose radii are 2·5 and 5 centimetres respectively. Draw from O a tangent to the smaller circle. Measure this tangent, and the length of the chord which the greater circle cuts off the tangent when produced.

8. Construct a triangle whose sides are respectively 8·5, 6·8, and 9·4 centimetres. Inscribe a circle in it and measure the radius of the circle.

9. If the square described on one side of a triangle is equal in area to the sum of the squares described on the other two sides, show that these two sides are at right angles to each other.

From one angle of an equilateral triangle a perpendicular is drawn upon the opposite side. Show that the square described on this perpendicular is equal to three-fourths of the square described on a side of the equilateral triangle.

10. Prove that the opposite angles of any quadrilateral figure inscribed in a circle are together equal to two right angles.

The straight line BE, which bisects the angle at B of a quadrilateral ABCD inscribed in a circle, meets the circle at E, and AD is a diameter of the circle. The side CD is produced through D to F. Show that DE bisects the angle ADF.

Geometrical Drawing.

THURSDAY, DECEMBER 13, 1906. $3\frac{3}{4}$ TO $4\frac{1}{2}$ OR $4\frac{1}{4}$ TO 5.

[*Mathematical Instruments may be used.*]

[*The solutions must be numbered to correspond with the problems, and the lines by which each problem is solved must be shown. The exercises are to be worked in pencil only, and the marks will be awarded for neatness of execution.*]

Candidates will find it convenient to draw two straight lines, horizontal and vertical, dividing each sheet of paper into four equal spaces, as shown in the diagram. A figure can be placed in each division.

Two sheets of paper will be given to each candidate.

1. On opposite sides of the straight line AC construct triangles ABC and ADC, where AB = 1″, BC = $2\frac{1}{2}$″, CD = 3″, DA = $1\frac{1}{2}$″, and the diagonal AC = 3″. Draw a circle within the figure ABCD to touch each of the sides AB, BC, CD.

2. Construct a right-angled triangle, the hypotenuse of which is to be 4″ in length, and one of the remaining sides $2\frac{1}{2}$″. Measure the length of the third side.

3. Construct an equilateral triangle ABC on a base BC, 3″ in length. From A draw AD perpendicular to BC, and produce AD to E so that AE = 3″. Join EB. Measure the number of degrees in the angle ABE.

4. Draw a plain scale of $1\frac{1}{2}$″ to 1′ to show feet and inches, 3′ being the greatest distance shown.

5. Draw a straight line BD to represent 20′ to the scale of 8′ to 1″. On opposite sides of BD describe triangles ABD and CDB so that AB = CD = 13′ and BC = DA = 12′ to the same scale.

Measure the length of AC to the nearest foot.

[*The scale need not be drawn.*]

6. Draw two straight lines AOC, BOD to cut one another at right angles at the point O. Take OA = OC = $\frac{3}{4}$″.

With centre A and distance $1\frac{1}{2}$″ make arcs to cut BOD in B and D. With centres A, B, C, D describe four equal circles, each circle to touch at least two of the others.

[*This problem is to be a test of accurate drawing.*]

CAMBRIDGE JUNIOR, 1906.

FRIDAY, DECEMBER 14, 1906. 9 TO $11\frac{1}{2}$ OR $9\frac{1}{2}$ TO 12.

Geometry.

Candidates can pass in Geometry by doing sufficiently well in the A *part of this paper. Figures should be drawn accurately with a hard pencil. In questions* 2, 5, 7, *and* 8, all lines required must be shown clearly.

Answers to the questions marked A *are to be fastened together in one bundle; those to the questions marked* B *in another bundle.*

A.

1. Prove that the exterior angle of a triangle is equal to the sum of the two interior opposite angles.

ABC is a triangle having the side BC produced to D, and the exterior angle ACD is double of the interior opposite angle ABC. Prove that ABC is isosceles.

2. AB and CD are two straight lines of unequal lengths, which intersect at O, and AO = OB, CO = OD. Join AC and BD, and prove that AC is parallel to BD.

Make an angle AOB of 54° and find a point P, within the angle, 4 centimetres distant from OA and 3 centimetres from OB. Then draw a straight line QR, passing through P and terminated by OA and OB, such that QR is bisected at P. Prove the correctness of your construction by geometrical reasoning, and measure QR.

3. ABC is a triangle having the angle BAC a right angle. Prove that the square described on BC is equal to the sum of the square described on AB and AC.

A kite at P, flown by a boy at Q, is vertically over a point R on the ground, which forms a horizontal plane in which Q also lies. If PQ = 505′ and QR = 456′, calculate the height of the kite above the ground.

4. Prove that the line joining the centre of a circle to the middle point of any chord of the circle is perpendicular to the chord.

A chord of a given circle varies its direction while continuing to pass

through a given point. Prove that the locus of the middle point of the chord is a circle.

5. Draw a triangle ABC having AB = 5·1″. BC = 3·8″, CA = 4·3″. Then describe the inscribed and circumscribed circles. Measure the angle between the lines joining A to the centres of the two circles.

B.

6. Illustrate and explain by means of a figure the geometrical theorem corresponding to the identity,—

$$a^2 - b^2 = (a + b)(a - b).$$

If E is the middle point of the side BC of a triangle ABC and AD is drawn perpendicular to BC meeting BC or BC produced in D, prove that the difference of the squares on AB, AC is equal to twice the rectangle contained by BC, DE.

7. Inscribe a regular hexagon in a circle of 3″ radius.

If A, B, C, D are four consecutive vertices of the hexagon so drawn, find, without using a protractor, the magnitude of the acute angle between the chords AC and BD. Show clearly how you arrive at your answer.

8. Make an angle of 64°. On one arm of the angle take two points distant respectively 1″ and 3″ from the apex, and through them draw the circle which touches the other arm of the angle. Measure the radius of the circle so drawn.

9. Prove that any line drawn parallel to one side of a triangle divides the other two sides, or those sides produced, proportionally.

ABCD is a quadrilateral with diagonals AC, BD. Through O, any point on AB, OP is drawn parallel to BD meeting AD in P, and OQ is drawn parallel to BC meeting AC in Q. Prove that if PQ is joined, PQ is parallel to CD.

10. Prove that the areas of similar triangles are in the duplicate ratio of the corresponding sides.

Each of the three sides of a triangle is trisected. Three lines are drawn, each joining two points of a section nearest an angle. These lines together with the middle segments of the sides form a hexagon ; find the ratio of its area to the area of the triangle.

FRIDAY, DECEMBER 14, 1906. $6\frac{1}{2}$ TO $7\frac{1}{2}$.

Geometrical Drawing.

[*Mathematical instruments may be used.*]

[*The solutions must be numbered to correspond with the problems, and the lines by which each problem is solved must be shown. The words "protractor," or "set square," must be written against those problems for which these instruments have been used. Marks are awarded for precision of drawing.*]

Candidates will find it convenient to draw two straight lines, horizontal and vertical, dividing each sheet of paper into four equal spaces, as shown in the diagram. A figure can be placed in each division.

Two sheets of paper will be given to each candidate.

1. Find a mean proportional between 1·95″ and 2·4″. On this mean proportional as base describe a triangle, the vertical angle of which is 60° and one of the remaining sides 1·8″. Measure the length of the remaining side of the triangle.

2. From a point A draw two straight lines AB, AE including an angle of 40°. These represent two straight roads meeting at A. Distances are measured at 10 furlongs to the inch. [A scale need not be drawn.] Take AB = 29½ furlongs. Find two points C and D in AE each 20 furlongs from B. Find the number of furlongs in AC and AD, and the number of degrees in the angles ACB and ADB.

3. Construct a right-angled triangle, the sides including the right angle being respectively 1½″ and 2½″. Describe a circle on the straight line joining the right angle to the middle point of the opposite side as diameter, and another circle to touch the least side of the right-angled triangle and its other two sides produced.

4. Draw a straight line of length 4¼″, to represent 7 kilometres. Construct on this line a plain scale showing kilometres and fifth parts of kilometres to measure up to 7 kilometres.

Draw a straight line 0·98 of an inch in length. If it be given that this line represents one mile, calculate the representative fraction, and use your scale to find the equivalent of one mile expressed in kilometres to the nearest fifth part of a kilometre.

5. Describe a regular pentagon about a circle the diameter of which is 2·87″. Circumscribe the pentagon by a circle.

6. Describe a circle of radius 1¾″. Divide this circle into three equal sectors. Within each of these sectors inscribe a circle.

[*This problem is to be a test of accurate drawing.*]

CAMBRIDGE SENIOR, 1906.

FRIDAY, DECEMBER 14, 9 TO 11 OR 9½ TO 11½.

Geometry.

N.B.—*All construction lines should be clearly shown. Attention is called to the alternative questions A, B, C, D at the end of the paper.*

1. Prove that two triangles, which have two sides of the one equal to two sides of the other, each to each, and also the angles contained by those sides equal, are congruent.

Two triangles ABC, DEF have the sides AB, AC equal to DE, DF respectively, and the angle ABC equal to the angle DEF. Investigate in each of the following three cases whether the triangles are necessarily congruent or not: (i) when AC is less than AB; (ii) when AC is greater than AB; (iii) when ACB is a right angle.

2. Construct a quadrilateral ABCD having each of the sides AB, BC, CD equal to 4 centimetres, and each of the angles BAD, CDA equal to 70°.

If AC, BD intersect at O, give a theoretical proof that BO is equal to CO.

3. Find the locus of a point which is equidistant from two intersecting straight lines; and make use of your result to find the centres of all the circles which touch the three sides of a given triangle.

4. Prove that chords of a circle which are equidistant from the centre are equal to one another.

Show how to draw chords of a given circle which are of given length.

and pass through a given point; and state within what limits the given length must lie for the construction to be possible (i) when the given point is within, and (ii) when it is without, the given circle.

5. Show how to find by a geometrical construction a mean proportional between two given straight lines.

Make the construction for lines whose lengths are 1·5 and 2·5″ respectively, and by measuring the length of the mean proportional determine as accurately as you can the value of $\sqrt{15}$.

6. Construct a regular hexagon about a circle whose radius is 5 centimetres. Measure the length of a side of this hexagon, and hence calculate its area.

7. If two triangles have an angle of the one equal to an angle of the other, and the sides about those equal angles proportional, prove that the triangles are similar.

A straight line OPAQ is drawn through the centre O of a circle, cutting the circumference in A, and the rectangle OP . OQ is equal to the square on OA. Prove that, if R is any point on the circumference of the circle, RA bisects the angle PRQ.

8. If three planes intersect each other in pairs, prove that their common sections either meet in one point or are parallel to one another.

Show how to draw a straight line parallel to a given straight line so as to intersect each of two other given straight lines which are not in the same plane.

9. On any chord OP of a given sphere, through a fixed point O on its surface, a point Q is taken so that the rectangle OP, OQ is equal to the square on the radius of the sphere. Prove that the locus of Q is a plane which bisects that radius of the sphere which passes through O.

N.B.—*One or more of the following questions A, B, C, D may be taken instead of an equal number of the questions* 6—9, *but lower marks will be assigned to them.*

A. Prove that a triangle which has two equal angles is isosceles.

In a quadrilateral ABCD the side AB is equal to the side CD, and the angle ABC to the angle BCD. Prove that the angle BAD is equal to the angle ADC.

B. Construct a triangle having its three angles in arithmetical progression, and its longest and shortest sides equal to 3″ and 2″ respectively. Measure its angles with your protractor, and state their values.

C. If two circles touch one another internally, show that the straight line which joins their centres will pass through the point of contact.

D. Prove that the opposite angles of a cyclic quadrilateral are supplementary.

Prove also, conversely, that, if the opposite angles of a quadrilateral are supplementary, a circle can be described about the quadrilateral.

THE END

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